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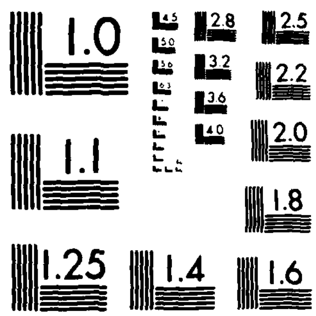

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OPTIMIZATION OF STOCHASTIC DYNAMIC SYSTEM WITH RANDOM COEFFICIENTS

by

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ABSTRACT

The problem of optimization of stochastic dynamic systems with random coefficients is discussed. Systems with both Wiener processes and uncertain random-process disturbances are dealt with, ~~in this report~~ and these include certain bilinear stochastic systems. It is the purpose ~~of this report~~ to study the optimal control and, to some extent, state estimation of such bilinear stochastic systems. By means of the stochastic Bellman equation, the optimal control of stochastic dynamic models with observable and unobservable coefficients is derived.

The stochastic-system model considered is the observable system with random coefficients that are a function of the solution of a certain unobservable Markov process with information data. Under the assumptions that the solution of the stochastic differential equation for the dynamic model involved in the problem formulation results in an admissible control and that the measurable information of all random parameters depend on the conditional-mean estimate to the unobservable stochastic process, the optimal control is a linear function of the observable states and a nonlinear function of random parameters.

The theory is then applied to an optimal-control design of an aircraft landing with a bad weather situation, to the control problem of longitudinal motion of an aircraft in wind gust, and to nonlinear filtering and tracking of the maneuvering target. The flare path compared with the desirable exponential-linear path provides a safe and comfortable landing for the optimal-control policy. Using the decoupled feedback law of longitudinal aircraft motion, it is shown that optimal-control policies for the elevator control angle and the aileron control angle is synthesized with the attack angle, the orientation rate of the aircraft and an unknown random parameter. In the

final example, a maneuvering target's state is estimated. Here, a bilinear stochastic model is assumed such that discrete velocity changes are at random times.

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OPTIMIZATION OF STOCHASTIC DYNAMIC SYSTEM WITH RANDOM COEFFICIENTS

1. INTRODUCTION

Most of the results in stochastic dynamic control and filtering theory were obtained with the assumptions that the process under some special considerations satisfy stochastic differential equations. Unfortunately, very little is known about optimal control theory for processes governed by general stochastic differential equations. In this area, most of the contributions are made by Fleming [1,2,3,4], Kushner [5,6], Wonham [7,8,9], Balakrishnan [10,11], Benes [12,13], Rishel [14,15], Bismut [16,17,18], Davis [19,20,21,22,23], Eliott [24,25], Haussman [26,27], Variya [28], Kolodziej [29], and Mohler and Kolodziej [30,31].

The status of continuous-time stochastic theory was summarized in Fleming's 1969 survey paper [3] and also introduced in his book [4] concerning the control of completely observable diffusion processes. In regard to problems with partial observation, possibly the most significant result was from Wonham's formulation of the separation principle [7] using the stochastic version of Bellman's dynamic programming which Wonham proved by reformulating the problem as one of the complete observations with the state being the conditional mean estimate produced by the Kalman-Bucy filter.

The dynamic-programming method is a useful approach in stochastic control. However, these conditions under dynamic programming are so much weaker than those required in deterministic control.

The dynamic-programming approach, while successful in many applications, suffers from many limitations. An immediate one is that the controls have to be smooth functions of the state in order that the stochastic differential equation has a solution in the Ito sense.

In problems of engineering design, it is often necessary to choose component values from an admissible set in addition to choosing control from an admissible class, such that the final product has the best performance in some appropriate sense. A typical example is the stochastic dynamic system with random coefficients which has attracted the attention of numerous authors [9,15,16,17,18,29,30,31,32,33,34,35,36,37]. One of the most formal approaches to stochastic control of such systems was presented by Bismut on convex analysis. Bismut solved a very general class of linear, quadratic, finite-dimensional, stochastic-control problems with random coefficients in which both state and control dependent noises are admitted, and he also discussed the necessary condition of stochastic optimality. Bensoussan and Voit [38] have considered optimal control problems for the stochastic evolution equation with deterministic operator-valued coefficients and have developed a separation principle for the problem. Recently, Ahmed [35] has solved the optimal-control porblem of the stochastic evolution equation and has considered a random operator Riccati equation and backward stochastic evolution equation. For the stochastic discrete-time system, the linear-quadratic optimal control has been investigated by Athans, et al., [36] and Ku, et al., [37].

Kolodziej [29], Mohler and Kolodziej [30,31] presented an alternate approach to the stochastic control of a class of linear stochastic systems with random coefficients. Kolodziej solved the problem of optimal control using dynamic programming. He also discussed the separation of filtering and control which was proven by reformulating the problem as one of complete observations and the control problem as an optimal regulator, which is a linear function of the unobservable part of the process and a nonlinear function of the observable parts. Sufficient conditions for optimal control were expressed through the existence of a bounded solution to a certain Cauchy problem for a parabolic type of partial differential equation. He assumed that the random process is conditionally Gaussian, i.e., the conditional distributions on the given random process are Gaussian (P-a.s.).

This dissertation presents the problem of the optimal control of stochastic differential equations with random coefficients. Systems with both Wiener processes and uncertain random process disturbances are dealt with in this study, and these include certain bilinear stochastic systems [31,39,40,41]. It is the objective of this thesis to study the optimal control and, to some extent, state estimation of such bilinear stochastic systems.

In general, it is difficult for a natural stochastic system to keep the conditional-Gaussian distribution of some states. Hence, Chapter 2 of this thesis considers observable stochastic-control systems with observable finite-dimensional random coefficients. Here, the stochastic Bellman equation to diffusion processes is

used to find the optimal-control law of the given system. This study results in a similar form of the optimal-control law [31].

The system model considered in Chapter 3 is the observable stochastic system with random coefficients that are a function of the solution of a certain unobservable Markov process with the information data. The results are shown for the suboptimal control of the above special stochastic model. All random parameters of the approximate models are replaced by the other random parameters that depend on the conditional-mean estimate of the Kalman-Bucy filter to the unobservable stochastic process. It is assumed that the solution of the stochastic differential equations in all cases results in an admissible control. The results for the problem are similar to optimal control with a Ricatti-like equation, but the methodology and the system models are somewhat different.

The application of the theoretical results of this thesis to physical systems is presented. The pseudo-Wiener process is generated by a Bernoulli time series using pseudo-uniform random numbers and pseudo-Gaussian random numbers $N(0,1)$. The numerical solution to a nonlinear partial differential equation is found by the method of lines [42], and the simulation studies result in the optimal control to simple stochastic systems. An aircraft landing model in a gusty wind [34,43,44,45] is studied here. The uncertain quantities of the landing model may be presented by the wind gust of the Dryden model [46]. The simulation results suggest the altitude of the aircraft during the landing period. The flare path compared with the desirable exponential-linear path provides a safe

and comfortable landing for the optimal control policy. The longitudinal motion of aircraft in a gusty wind given in Chapter 4 can be obtained by theoretical results of Chapter 3.

The decoupled feedback law of longitudinal motion of aircraft which included unknown quantities and is also subject to uncertain noise is discussed. Using an approximate stochastic model of longitudinal aircraft motion for the worst situation, it is shown that optimal control policies for the elevator control angle and the aileron control angle is synthesized with the attack angle, the orientation rate of aircraft, and an unknown random parameter. The simulation results are compared with the assumptions of observable and unobservable cases, respectively.

In Chapter 5, anti-submarine target-motion analysis on nonlinear filtering and tracking is presented. A common method in the target-tracking problem [47,48,49,50,51,52] is to model the target dynamics in a rectangular-coordinate system which results in a linear set of state equations because of the assumption that drag forces are linear relationships in states or are neglected. However, the drag forces are proportional to velocities squared in each target motion's directions [53,54]. Hence, a more appropriate mathematical model is derived for the consideration of nonlinear target dynamics underwater. With this model, this chapter compares performances between an extended Kalman filter and a truncated second-order nonlinear filter as applied to bearing-only-target tracking. In the existence of a maneuvering of target, many authors [47,48,49,50,55, 56,57,58,59,60,61,62,63] have proposed compensators for the Kalman

filter for adaptation to maneuvering situation. The target motion equation for estimates of the vehicle maneuvering performance may be represented by a bilinear stochastic system which has jumps so that between jumps it remains in specific states at random time T. The stochastic process involved in the maneuver is discussed. The last section shows the filtering and control problem as the adaptive control is a function of random time T and the new extended states by solving certain Riccati equations.

Notation

The following notations will be used throughout:

R^n	Euclidean n-dimensional space
C_T	Space of continuous functions on $[0,T]$
*	Transposition of a vector or a matrix
tr	Trace of a matrix
$\Vert\cdot\Vert$	Euclidean norm
$a \times b$	Direct product of a and b
$[B]_{ij}$	i,jth elements of a matrix B
$[b]_i$	ith element of a vector b
K, k_i	Positive constants
$x \in A$	x is an element of A
$[a,b]$	Closed interval
$\frac{\partial}{\partial \xi i}$	Gradient vector of nonanticipative functionals
$\frac{\partial^2}{\partial \xi_i \partial \xi^*_j}$	Jacobian matrix of nonanticipative functionals
(Ω, F, P)	Complete probability space
$x(t)$	Value at a particular elementary event $\omega \in \Omega$
$\{x(t)\}$	Stochastic vector process
$E(x_t)$	Expectation of x_t
$E(x_t \vert y_t)$	Conditional expectation with respect to a given observation measure y_t
$Cov(x_s x_t)$	Covariance of x_s and x_t
$Var\ (x_s)$	Variance of x_s
x_t^n	Sequence $n=1,2,\ldots,$

$\|A\|$ The Euclidean norm is defined as A being a vector or a matrix as follows

$$\|A\|^2 = tr(AA^*);$$

F_t Sub-σ-field of F

$\mathcal{Y}_t$ σ-algebra $\{y_s, 0 \leq s \leq t\}$ generated by observable stochastic process $\{y_t, t \in [0,T]\}$

$\mathcal{Z}_t$ σ-algebra $\{x_s$ and $y_s, 0 \leq s \leq t\}$ generated by the observable stochastic process $\{x_t$ and $y_t, t \in [0,T]\}$

$B(t,\cdot)$ Measurable nonanticipative functional parameter

inf infimum

$\dot{x}$ Denotes $\frac{dx}{dt}$

2. OPTIMAL CONTROL OF A STOCHASTIC SYSTEM WITH RANDOM COEFFICIENTS

Let (Ω,F,P) be a complete probability space; F_t, an nondecreasing family of sub-σ-algebras of F, $t \in [0,T]$; $\{w_t\}$, an F_t-adapted Wiener process of dimension ℓ. Let x_t and z_t, $t \in [0,T]$, (dimensions n and m, respectively), be observable continuous processes satisfying:

$$d\,x_t = A(t,z_t)x_t\,dt + B(t,z_t)u_t\,dt + G(t,z_t)dw_t, \tag{2-1}$$

$$d\,z_t = C(t,z_t)dt + D(t,z_t)dw_t, \tag{2-2}$$

$$x(0) = x_0, \quad z(0) = z_0.$$

The random variables x_0 and z_0 are assumed to be independent of w_t, $t \in [0,T]$. Let F_t be the σ-algebra in the finite space C_T of the continuous finite functions, $\theta = \{\theta_s, s \le t\}$, $t \le T$. Each of the functionals $A(t,\Theta)$, $B(t,\Theta)$, $G(t,\Theta)$, $C(t,\Theta)$, $D(t,\Theta)$ is assumed to be measureable with respect to F_t and have the dimensions $n \times n$, $n \times p$, $n \times \ell$, $m \times 1$, $m \times \ell$, respectively.

The control u_t is assumed to be of a Markov type, i.e., $u_t = u(x_t,z_t,t)$. Then, the problem is to find a control in an admissible control set $\mathcal{U}$ that minimizes the average cost functional

$$J(u) = E[\int_0^T L(t,x_t,u_t)dt]. \tag{2-3}$$

2.1 Optimal Control with Complete Measurement Information

If the stochastic differential equation (2-1) and (2-2) satisfy the assumptions of the theorem 2.1, then the unique solution of (2-1)

and (2-2) exists [47,48].

<u>Theorem 2.1</u> Let $A(s,\eta)$, $B(s,\eta)$, $G(s,\eta)$, $C(s,\eta)$, $D(s,\eta)$, $s \in [0,T]$ be F_t-measurable functionals, $C(s,\eta)$, $D(s,\eta)$ satisfy the Lipschitz condition, and $|A(t,\eta)| \le k_1 < \infty$, $|B(t,\eta)| \le k_2 < \infty$, $\int_0^T \|G(t,\eta)\|^2 \le k_3 < \infty$. Then, if $\begin{bmatrix} x_0 \\ z_0 \end{bmatrix}$ is a F_0-measurable random vector, $E[x_0^* x_0] + E[z_0^* z_0] < \infty$, the stochastic differential equations (2-1) and (2-2) have the unique solutions, i.e., x_t, z_t are F_t-adapted and x_t, z_t are given by

$$x_t = x_0 + \int_0^t A(s,z_s)\, x_s\, ds + \int_0^t B(s,z_s) u_s ds + \int_0^t G(s,z_s) dw_s, \tag{2-4}$$

$$z_t = z_0 + \int_0^t C(s,z_s) ds + \int_0^t D(s,z_s) dw_s \tag{2-5}$$

The above integrals are defined in the Ito sense. Here k_1, k_2, k_3 are positive constants.

<u>Proof</u> The proof is omitted. The reader is referred for details to Liptser and Shiryayev [47,48]. ∎

Now consider the problem of optimal control of x_t for $t \in [0,T]$ based on the complete observations of x_t and z_t. The observable states x_t and z_t are the solution of equations (2-1) and (2-2), respectively. The problem is to choose a control law u_t so as to minimize the cost functional (2-3).

For the solution of the optimal control, the following assumptions are introduced [29,31]:

1) Assumptions of the theorem 2.1 are satisfied.
2) $L(t,x_t,u_t) = x_t^* Q(t,z_t) x_t + u_t^* R(t,z_t) u_t$,
   where for $\eta \in R^m$, $t \in [0,T]$, $Q(t,\eta)$ is a nonnegative
   definite matrix, and $R(t,\eta)$ is uniformly positive

definite, i.e., elements of its inverse are uniformly bounded measurable functions,.

3) The control $u_t \in \mathcal{U}$ satisfies

$$\int_0^T E[\|u_t\|^2]\, dt < \infty,$$

$$u_t = u(t, x_t, z_t),$$

and is such that (2-1) has a unique solution.

Let $s \in [0,T]$ be the initial time; $x_s = x_0$, the initial state, $u_t \in \mathcal{U}$, and x_t, the corresponding response of the system (2-1). The conditional remaining cost on the time s = 0 is defined by

$$W^u(s, x_s, z_s) = E[\int_0^T L(t, x_t, z_t, u_t)dt \mid x_s = x_0,\ z_s = z_0], \quad (2\text{-}6)$$

as the expected cost corresponding to the control u_t and initial state x_0 and z_0. Here, T is a fixed terminal time and $L(\cdot,\cdot,\cdot)$ is a bounded measurable function. The problem is to minimize J(u) on $\mathcal{U}$.

Let

$$\sigma(t,\eta) \triangleq \begin{bmatrix} G(t,\eta)\ G^*(t,\eta) & G(t,\eta)\ D^*(t,\eta) \\ D(t,\eta)\ G^*(t,\eta) & D(t,\eta)\ D^*(t,\eta) \end{bmatrix},$$

and assume that $\sigma(t,\eta)$ is uniformly positive definite over $t \in [0,T)$, $\eta \in R^m$, i.e.,

$$\sum_{i,j}^{n+m} \sigma_{ij}\ (t,\eta) y_i y_i \geq k|yy^*|, \quad k > 0,$$

for all $y \in R^{n+m}$. This essentially states that noise enters every component of (2-1), whatever the coordinate system.

Define

$$V(t,y) = \inf_u V^u(t,y),$$

where $V^u(t,y) = W^u(s,x_t,z_t)$, $y = \begin{bmatrix} x_t \\ z_t \end{bmatrix}$, $(t,y) \in (0,T] \times R^{n+M}$.

From the above assumptions it follows that [Fleming [4]]

$$V_s + A^u(s)V + L(s,y,\ u(s,y)) \geq 0,$$

where

$$A^u(s)V \triangleq \frac{1}{2} \sum_{i,j=1}^{n+m} \sigma_{ij} \frac{\partial^2}{\partial y_i \partial y_i^*} V + \sum_{i=1}^{n} (A(s,\eta)\zeta$$

$$+ B(s,\eta)u_s)_i \frac{\partial}{\partial y_i} V + \sum_{i=n+1}^{n+m} (C(s,\eta))_i \frac{\partial}{\partial y_i} V, \ \eta \in R^m,$$

$$\zeta \in R^n$$

and

$$V_s \triangleq \frac{\partial V}{\partial s}.$$

The above equality holds if $u = u^o(s,y)$, where u^o is an optimal feedback control law. This leads to the continuous-time dynamic-programming equation:

$$V_s + \min_u (A^u(s)V + L(s,y,u)) = 0, \quad V(T,y) = 0. \tag{2-7}$$

Theorem 2.2 Assume that the value function V satisfying the stochastic Bellman equation (2-7) exists and is differentiable in (t,y). If a control $u_t^o \in \mathcal{U}$ satisfies

$$A^{u^o}(t)\ V + L(t,y,u^o(s,y)) \leq A^u(t)\ V + L(t,y,u(t,y)), \tag{2-8}$$

for all $u_t \in \mathcal{U}$, $(t,y) \in [0,T] \times R^n \times R^m$,

then u_t^o is an optimal control.

<u>Proof</u> Let $u_t \varepsilon \mathcal{U}$ be any control and $y_t = \begin{bmatrix} x_t \\ z_t \end{bmatrix}$ the corresponding trajectory. From Bellman's principle of optimality and the Taylor expansion on V,

$$V(t,x_t,z_t) = \min_{u_t \varepsilon \mathcal{U}} \{L(t,x_t,z_t,u_t)\delta + V(t,x_t,z_t) + \frac{\partial V}{\partial t}(t,x_t,z_t)\delta + A^u(t)\, V(t,x_t,z_t) + O(\delta)\},$$

where any $\delta \varepsilon (0, T-t)$. Now dividing by δ and let $\delta \to 0$, then

$$\frac{\partial V}{\partial t}(t,x_t,z_t) + \min_{u_t \varepsilon \mathcal{U}} (A^u(t)V(t,x_t,z_t) + L(t,x_t,z_t,u_t) = 0 \quad (2\text{-}9)$$

$$V(T,x_T,z_T) = 0.$$

Equation (2-9) is the Bellman-Hamilton-Jacobi, BHJ, equation derived heuristically above (2-7). Then from (2-9)

$$\frac{\partial V}{\partial t}(t,x_t,z_t) + A^u(t)V(t,x_t,z_t) + L(t,x_t,z_t,u_t) \geq 0.$$

Thus, using Ito's formula it follows that

$$\frac{d}{dt}(V(t,x_t,z_t)) = \frac{\partial V}{\partial t}(t,x_t,z_t) + A^u(t)\; V(t,x_t,z_t) \geq -L(t,x_t,z_t,u_t). \quad (2\text{-}10)$$

Taking the expectation of the both sides of (2-10) and integrating, the following equation is obtained.

$$E[V(T,x_T,z_T) - V(0,x_0,z_0)] = E[\int_0^T \frac{dV}{dt}(t,x_t,z_t)dt \geq -E\int_0^T L(t,x_t,z_t,u_t)dt. \quad (2\text{-}11)$$

Since $V(T,x_T,z_T) = 0$, this shows that the following inequality holds:

$$\mathcal{E}[V(0,x_0,z_0)] \leq \mathcal{E} \int_0^T L(t,x_t,z_t,u_t)dt = J(u_t). \tag{2-12}$$

With u_t^o, the same calculations apply to give

$$\mathcal{E}[V(0,x_0,z_0)] = J(u_t^o) \tag{2-13}$$

But now (2-11) and (2-13) show that u_t^o is optimal; hence the above results imply that

$$J(u_t^o) \leq J(u_t), \quad \text{for all } u_t \in \mathcal{U}. \quad \blacksquare \tag{2-14}$$

Assume that the value function $V(t,x_t,z_t)$ on $[0,T] \times R^n \times R^m$ has the following from [29,31]:

$$V(t,x_t,z_t) = x_t^* \Lambda_1(t,z_t)x_t + \Lambda_2(t,z_t)x_t + \Lambda_3(t,z_t), \tag{2-15}$$

where Λ_1, Λ_2, and Λ_3 are the solutions of certain nonlinear partial differential equations and $x_t \in R^n$, $z_t \in R^m$. Then, the stochastic BHJ equation is given by

$$\begin{aligned}\frac{\partial}{\partial t} (x_t^* \Lambda_1(t,z_t)x_t + \Lambda_2(t,z_t)x_t + \Lambda_3(t,z_t)) \\ + \min_{u_t \in \mathcal{U}} (A^u(t)(x_t^* \Lambda_1(t,z_t)x_t + \Lambda_2(t,z_t)x_t \\ + \Lambda_3(t,z_t)) + L(t,x_t,z_t,u_t)) = 0 .\end{aligned} \tag{2-16}$$

It is enough to show that we can find Λ_1, Λ_2, Λ_3 and u_t^o that satisfies (2.16). This is shown in the theorem below.

<u>Theorem 2.3</u>

The optimal control u_t^o, $t \in [0,T]$, exists and is given by

$$u_t^{o} = -R^{-1}(t,z_t)B^*(t,z_t)(\Lambda_1(t,z_t)x_t + \frac{1}{2}\Lambda_2(t,z_t)), \qquad (2\text{-}17)$$

if there exist the nonnegative definite symmetric matrices Λ_1 and Λ_2 satisfying the following nonlinear partial differential equations:

$$\dot{\Lambda}_1 + A^*\Lambda_1 + \Lambda_1 A + Q - \Lambda_1 BR^{-1}B^*\Lambda_1 + C^*\frac{\partial}{\partial z_t}\Lambda_1 + \frac{1}{2}\,\mathrm{tr}(DD^*\frac{\partial^2}{\partial z_t \partial z_t^*})\Lambda_1 = 0, \qquad (2\text{-}18)$$

$$\dot{\Lambda}_2 + \Lambda_2 A - \Lambda_2 BR^{-1}B^*\Lambda_1 + C^*\frac{\partial}{\partial z_t}\Lambda_2 + \frac{1}{2}\,\mathrm{tr}(DD^*\frac{\partial^2}{\partial z_t \partial z_t^*})\Lambda_2 + 2(GD^*\frac{\partial}{\partial z_t})^*\Lambda_1 = 0,$$

$$\Lambda_1(T,z_T) = 0, \quad \Lambda_2(T,z_T) = 0, \quad z_t \in R^m,$$

(the argument (t,z_t) is omitted for bervity).

<u>Proof</u>. Using the dynamic-programming equation (2-16), it follows that

$$x_t^*\frac{\partial\Lambda_1}{\partial t}x_t + \frac{\partial\Lambda_2}{\partial t}x_t + \frac{\partial\Lambda_3}{\partial t} + \min_{u_t\in U}\Big(\frac{1}{2}\Big(\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\sigma_{ij}(\frac{\partial}{\partial x_t})_i(\frac{\partial}{\partial x_t})_j$$

$$+\sum_{j=n+1}^{n+m}\sigma_{ij}(\frac{\partial}{\partial x_t})_i(\frac{\partial}{\partial x_t})_j\Big) + \sum_{i=n+1}^{n+m}\Big(\sum_{j=1}^{n}\sigma_{ij}(\frac{\partial}{\partial z_t})_i(\frac{\partial}{\partial x_t})_j$$

$$+\sum_{j=n+1}^{n+m}\sigma_{ij}(\frac{\partial}{\partial z_t})_i(\frac{\partial}{\partial z_t})_j\Big)\Big)\cdot(x_t^*\Lambda_1 x_t + \Lambda_2 x_t + \Lambda_3) \qquad (2\text{-}19)$$

$$+\sum_{i=1}^{n}(Ax_t + Bu_t)_i(\frac{\partial}{\partial x_t})_j\cdot(x_t^*\Lambda_1 x_t + \Lambda_2 x_t + \Lambda_3)$$

$$+\sum_{i=1}^{m}(C)_i(\frac{\partial}{\partial z_t})_i(x_t^*\Lambda_1 x_t + \Lambda_2 x_t + \Lambda_3) + x_t^*Qx_t + u_t^*Ru_t) = 0.$$

The equation (2-19) can be represented by the following equation:

$$x_t^{*}\dot{\Lambda}_1 x_t + \dot{\Lambda}_2 x_t + \dot{\Lambda}_3 + \frac{1}{2}\Big(\Big(\sum_{i,j=1}^{n} \sigma_{ij}\Big(\frac{\partial}{\partial x_t}\Big)_i\Big(\frac{\partial}{\partial x_t}\Big)_j +$$

$$\sum_{i,j=n+1}^{n+m} \sigma_{ij}\Big(\frac{\partial}{\partial z_t}\Big)_i\Big(\frac{\partial}{\partial z_t}\Big)_j\Big) \cdot (x_t^{*}\Lambda_1 x_t + \Lambda_2 x_t + \Lambda_3) +$$

$$\Big(\sum_{i=1}^{n}\sum_{j=n+1}^{n+m} \sigma_{ij}\Big(\frac{\partial}{\partial x_t}\Big)_i\Big(\frac{\partial}{\partial z_t}\Big)_j + \sum_{i=n+1}^{n+m}\sum_{j=1}^{n} \sigma_{ij}\Big(\frac{\partial}{\partial z_t}\Big)_i\Big(\frac{\partial}{\partial x_t}\Big)_j\Big)$$

$$\cdot(x_t^{*}\Lambda_1 x_t + \Lambda_2 x_t)\Big) + \sum_{i=1}^{m} (C)_i\Big(\frac{\partial}{\partial z_t}\Big)_i(x_t^{*}\Lambda_1 x_t + \Lambda_3) + x_t^{*}Q x_t$$

$$+ \min_{u_t \in \mathcal{U}}\Big(\sum_{i=1}^{n} (Bu_t)_i\Big(\frac{\partial}{\partial x_t}\Big)_i(x_t^{*}\Lambda_1 x_t + \Lambda_2 x_t) + u_t^{*} R u_t\Big) = 0 . \tag{2-20}$$

Only the last term in (2-20) includes the control, and

$$\sum_{i=1}^{n} (Bu_t)_i\Big(\frac{\partial}{\partial x_t}\; (x_t^{*}\Lambda_1(t,z_t)x_t + \Lambda_2(t,z_t)x_t) + u_t^{*}R(t,z_t)u_t\Big)$$

$$= (u_t + R^{-1}(t,z_t)(B^{*}(t,z_t)\Lambda_1(t,z_t)x_t + \frac{1}{2} B^{*}(t,z_t)\Lambda_2(t,z_t)))^{*}$$

$$\cdot R(t,z_t)(u_t + R^{-1}(t,z_t)(B^{*}(t,z_t)\Lambda_1(t,z_t)x_t + \frac{1}{2} B^{*}(t,z_t)\Lambda_2(t,z_t)))$$

$$- (R^{-1}(t,z_t)(B^{*}(t,z_t)\Lambda_1(t,z_t)x_t + \frac{1}{2} B^{*}(t,z_t)\Lambda_2(t,z_t))^{*}$$

$$\cdot R^{-1}(t,z_t)(B^{*}(t,z_t)\Lambda_1(t,z_t)x_t + \frac{1}{2} B^{*}(t,z_t)\Lambda_2(t,z_t)) . \tag{2-21}$$

It follows that the minimum is obtained at

$$u_t^{o} = -R^{-1}(t,z_t)B^{*}(t,z_t)(\Lambda_1(t,z_t)x_t + \frac{1}{2}\Lambda_2(t,z_t)) . \tag{2-22}$$

Substituting (2-22) into (2-20) gives

$$x_t^*\Big(\frac{\partial}{\partial t}\Lambda_1 + A^*\Lambda_1 + \Lambda_1 A + Q + C^*\frac{\partial^2}{\partial z_t}\Lambda_1 + \frac{1}{2}\,\mathrm{tr}\Big(DD^*\frac{\partial^2}{\partial z_t \partial z_t^*}\Big)\Lambda_1$$

$$- \Lambda_1 BR^{-1}B^*\Lambda_1\Big)x_t + \Big(\frac{\partial}{\partial t}\Lambda_2 + \Lambda_2 A + C^*\frac{\partial}{\partial z_t}\Lambda_2 + \frac{1}{2}\,\mathrm{tr}\Big(DD^*\frac{\partial^2}{\partial z_t \partial z_t^*}\Big)\Lambda_2$$

$$+ 2\Big(GD^*\frac{\partial}{\partial z_t}\Big)^*\Lambda_1 - \Lambda_2 BR^{-1}B^*\Lambda_1\Big)x_t + \Big(\frac{\partial}{\partial t}\Lambda_3 + C^*\frac{\partial}{\partial z_t}\Lambda_3$$

$$+ \frac{1}{2}\,\mathrm{tr}\Big(DD^*\frac{\partial^2}{\partial z_t \partial z_t^*}\Big)\Lambda_3 + \mathrm{tr}(GG^*)\Lambda_1\Big) = 0. \qquad (2\text{-}23)$$

Because (2-23) has to be established for all $x_t \in R^n$, Λ_1, Λ_2, Λ_3 must satisfy

$$\frac{\partial \Lambda_1}{\partial t} = -A^*\Lambda_1 - \Lambda_1 A - Q + \Lambda_1 BR^{-1}B^*\Lambda_1 - C^*\frac{\partial}{\partial z_t}\Lambda_1 - \frac{1}{2}\,\mathrm{tr}\Big(DD^*\frac{\partial^2}{\partial z_t \partial z_t^*}\Big)\Lambda_1,$$

$$\frac{\partial \Lambda_2}{\partial t} = -\Lambda_2 A + \Lambda_2 BR^{-1}B^*\Lambda_1 - C^*\frac{\partial}{\partial z_t}\Lambda_2 - \frac{1}{2}\,\mathrm{tr}\Big(DD^*\frac{\partial^2}{\partial z_t \partial z_t^*}\Big)\Lambda_2 - 2\Big(GD^*\frac{\partial}{\partial z_t}\Big)^*\Lambda_1,$$

$$\frac{\partial \Lambda_3}{\partial t} = -C^*\frac{\partial}{\partial z_t}\Lambda_3 - \frac{1}{2}\,\mathrm{tr}\Big(DD^*\frac{\partial^2}{\partial z_t \partial z_t^*}\Big)\Lambda_3 - \mathrm{tr}(GG^*)\Lambda_1,$$

and

$$\Lambda_1(T, z_T) = 0, \quad \Lambda_2(T, z_T) = 0, \quad \Lambda_3(T, z_T) = 0,$$

respectively. ∎

The solutions $\Lambda_1(t,\eta)$ and $\Lambda_2(t,\eta)$ to the above Cauchy problem can be shown to be nonnegative definite and uniformly bounded for all $(t,\eta) \in [0,T] \times R^m$ [29].

Mohler developed bilinear stochastic systems that are the

diffusion models for migration of people, biological cells, etc., [30,31,66,67,68]. The system equation in (2-1) is a class of coupled bilinear stochastic equations. In this particular case, the optimal control of the bilinear stochastic system of diffusion processes (2-1) and (2-2) is given by (2-17). The following examples belong to the class of coupled bilinear stochastic systems.

Example 2.1 If $A(t,z_t) = 0$, $G(t,z_t) = G(t)$, then the equation (2-1) is

$$dx_t = B(t,\cdot)u_t \, dt + G(t)dw_t, \quad x(0) = x_0 \in R^n; \tag{2-24}$$

where $B(t,\cdot)$ is composed of unknown coefficients. Such uncertain parameters may be regarded as additional state variables. These additional state variables with uncertain gain might be approximated by

$$dz_t = C(t,z_t)dt + D(t)dw_t, \quad z(0) = z_0 \in R^m. \tag{2-25}$$

If $B(t,\cdot) = B(t)z_t$, (2-24) is bilinear in z_t and u_t, and the system has an extended state with R^{n+m}. At this point the problem of uncertain parameter becomes a parameter-identification problem and the system equation is a bilinear stochastic differential equation. An aircraft landing process [44] may be represented by this type of bilinear stochastic equation.

Example 2.2

Consider the stochastic differential equation with the random coefficients,

$$dx_t = A(t,z_t)x_t\,dt + B(t,z_t)\,u_t dt + G(t)dw_t^{\,1}, \qquad (2\text{-}26)$$

where state $x_t \in R^n$ is observable and uncertain disturbance process $z_t \in R^m$ is partially observable; and

$$dz_t = C(t)z_t\,dt + D(t)\,dw_t^{\,2}$$

with the observation

$$dy_t = F(t)z_t\,dt + H(t)\,dw_t^{\,3}.$$

The problem of optimal control of (2-26) under the given information will be discussed in Chapter 3.

Comment: The optimal control of partially observable state x_t was discussed by Kolodziej [29]. There exists the optimal filter in the conditionally Gaussian case. The form of optimal control law of this particular case is the same as the equation (2-17).

The stochastic dynamic model in (2-26) may be approximated in certain ways for σ-algebra $\mathcal{Y}_t$ generated by observation $\{y_t, t \in [0,T]\}$ or σ-algebra $\mathcal{Z}_t$ generated by observation $\{x_t, y_t, t \in [0,T]\}$. If there is an exact solution to the problem of finite conditional estimate $E\{\begin{smallmatrix} x_t \\ z_t \end{smallmatrix} | \mathcal{Z}_t\}$, the stochastic dynamic equation may provide optimal control of (2-26). If $A(t,z_t)$ and $B(t,z_t)$ in (2-26) are replaced with $E[A(t,z_t)|\mathcal{Y}_t]$, $E[B(t,z_t)|\mathcal{Y}_t]$, the problem of sub-optimal control is similar to that introduced at the beginning of this chapter.

3. APPROXIMATE STOCHASTIC MODELS

Formulating a mathematical stochastic model for the dynamic behavior of a physical system is in terms of the evolution of the state x_t, $t \in [0,T]$, of the system which is a stochastic process defined on some complete probability space (Ω,F,P) under the influence of control u_t and disturbances z_t as the solution of the stochastic differential equation,

$$dx_t = A(t,z_t)x_t dt + B(t,z_t)u_t dt + G(t,z_t)dw_t^1, \; x(0) = x_0, \tag{3-1}$$

and x_t is an observable process and w_t^1 is a Wiener process. Assume that this stochastic system has disturbances as the solution of a stochastic differential equation

$$dz_t = C(t)z_t \, dt + D(t)dw_t^2 \, , \; z(0) = z_0, \tag{3-2}$$

with observation

$$dy_t = F(t)z_t dt + H(t)dw_t^3, \tag{3-3}$$

where z_t is unobservable and w_t^i, $i = 1,2,3$, are mutually independent Wiener processes of dimensions ℓ_i, respectively. The matrices A, B, G, C, D, F, and H have the dimensions $n \times n$, $n \times p$, $n \times \ell_1$, $m \times m$, $m \times \ell_2$, $k \times m$ and $k \times \ell_3$, respectively. Assume that $A(t,z_t)$, $B(t,z_t)$, and $G(t,z_t)$ of $(t,z_t) \in [0,T] \times R^m$ are Borel measurable.

Let Z_t be the σ-algebra generated by $\{x_s$ and y_s, $0 \leq s \leq t\}$. The control u_t of dimension P is assumed to be Z_t measurable for every $t \in [0,T]$. The control u_t is to be chosen so as to minimize the cost

$$J(u) = \mathcal{E}[\int_0^T x_t^* \, Q(t,z_t)x_t + u_t^* R(t,z_t)u_t]dt, \tag{3-4}$$

where the symmetric matrices Q and R have the dimensions n x n, and p x p, respectively.

Consider the mean-square estimate of z_t: $\mathcal{E}[z_t | \mathcal{Y}_t,\ 0 \leq t \leq T]$. $\mathcal{Y}_t$ denotes the σ-algebra generated by $\{y_s,\ 0 \leq s \leq t\}$. Let $\hat{z}_t$ be $\mathcal{E}[z_t | \mathcal{Y}_t,\ 0 \leq t \leq T]$, and $\mathcal{Y}_t \subseteq \mathcal{Z}_t$. Under the proper assumptions, the estimate $\hat{z}_t$ satisfies the linear stochastic equation given by

$$d\hat{z}_t = C(t)\ \hat{z}_t dt + \Gamma_t\ F(t)^* (\ H(t)\ H(t)^*)^{-1} d\ \nu_t, \tag{3-5}$$
$$z_0 = \mathcal{E}[z_0],$$

where $d\nu_t$ is the innovation process corresponding to (3-3), and Γ_t is the error-covariance matrix which satisfies the following matrix Riccati equation:

$$d\Gamma_t = (D(t)D^*(t) - \Gamma_t\ F^*(t)\ (H(t)H(t)^*)^{-1}\ F(t)\Gamma_t + C(t)\Gamma_t + \Gamma_t C^*(t))dt, \tag{3-6}$$
$$\Gamma_0 = \mathrm{Cov}[z_0].$$

The system of equations (3-5) and (3-6) have a unique solution for Γ_t in the class of symmetric nonnegative-definite matrices.

3.1 Simple Approximation to Stochastic Modeling

The stochastic system model in (3-1) has an unknown random coefficient which depends on the unobservable disturbance stochastic equation (3-2). For the first method consider the approximate stochastic model for (3-1).

$$dx_t \cong \bar{A}(t,\hat{z}_t,\Gamma_t)\, x_t\, dt + \bar{B}(t,\hat{z}_t,\Gamma_t)\, u_t\, dt + \bar{G}(t,\hat{z}_t,\Gamma_t) d\, w_t^{\,1}, \tag{3-7}$$

where

$$\begin{aligned}
\bar{A}(t,\hat{\xi},\Gamma_t) &= E[A(t,\xi)\,|\,Y_t,\ 0 \le t \le T] \\
&= \int_{-\infty}^{\infty} A(t,\xi)\cdot(1/(2\pi)^{\frac{m}{2}}|\Gamma_t|^{\frac{1}{2}})\cdot \mathrm{EXP}[-\tfrac{1}{2}(\xi-\hat{\xi})^{*}\Gamma_t^{-1}(\xi-\hat{\xi})]d\xi, \\
\bar{B}(t,\hat{\xi},\Gamma_t) &= E[B(t,\xi)\,|\,Y_t,\ 0 \le t \le T], \\
\bar{G}(t,\hat{\xi},\Gamma_t) &= E[G(t,\xi)\,|\,Y_t,\ 0 \le t \le T].
\end{aligned} \tag{3-8}$$

Here the mean estimate $\hat{z}_t$ is the solution of (3-5) and the covariance Γ_t is the solution of (3-6). The problem is to find the control u_t from the admissible class that minimizes the cost functions (3-4). For the linear-regulator problem, (3-4) is approximated by the following new cost functions:

$$\hat{J}(u) = E[\int_0^T (x_t^{\,*}\, \bar{Q}(t,\hat{z}_t,\Gamma_t)x_t + u_t^{\,*}\bar{R}(t,\hat{z}_t,\Gamma_t)u_t)dt], \tag{3-9}$$

where

$$\begin{aligned}
\bar{Q}(t,\hat{\xi}_t,\Gamma_t) &= \int_{R^m} Q(t,\xi_t)\cdot f(\hat{\xi}_t,\Gamma_t,\xi_t)d\xi_t, \\
\bar{R}(t,\hat{\xi}_t,\Gamma_t) &= \int_{R^m} R(t,\xi_t)\cdot f(\hat{\xi}_t,\Gamma_t,\xi_t)d\xi_t.
\end{aligned} \tag{3-10}$$

Here $f(\hat{\xi}_t,\Gamma_t,\xi_t)$, $\xi_t \in R^m$, $t \in [0,T]$, is the m-dimensional Guassian density function with mean $\hat{\xi}_t$ and covariance Γ_t. This transformation results in the case of the system (3-7), and the symmetric matrices $\bar{Q}$ and $\bar{R}$ have the dimensions n x n and p x p, respectively.

Remark: The distribution of the estimation error $\tilde{\xi} = \xi - \hat{\xi}$ is Gaussian, and $\bar{Q}\,(t,\hat{\xi},\Gamma_t)$ and $\bar{R}(t,\hat{\xi},\Gamma_t)$ may be calculated by (3-10).

$\hat{J}(u)$ has the similar form of $J(u)$.

Let $\mathcal{U}$ be a certain class of admissible controls. The control $u_t^o \varepsilon\, \mathcal{U}$ is called optimal for (3-7) if

$$\hat{J}(u_t^o) = \inf_{u_t \varepsilon \mathcal{U}} \hat{J}(u_t), \tag{3-11}$$

where inf is taken over the class of all admissible controls. One of the analyses of the above control problem is included to get the optimal-control law of the observable linear control system with quadratic criteria which has random coefficients being certain functionals of the Wiener process ν_t [19].

For the solution of the optimal control of (3-7), it will have the same assumptions as are made in Chapter 2 with the proper parameters of (3-7). Then, the unique strong solution of (3-7) exists because the control u_t, $0 \leq t \leq T$, is admissible if for this control, theorem 2.1 and assumptions 1)-3) in Chapter 2 is satisfied.

Assume that the value function is of the following form

$$V(t,x_t,\xi,\gamma) = x_t^* \Lambda_1(t,\xi,\gamma)x_t + \Lambda_2(t,\xi,\gamma)x_t + \Lambda_3(t,\xi,\gamma), \tag{3-12}$$
$$x_t \varepsilon\, R^n, \xi \varepsilon\, R^m,\ \gamma \varepsilon\, R^m \times R^m,$$

where Λ_1, Λ_2 and Λ_3 are symmetric matrices which satisfy a certain nonlinear partial differential equations which will be discussed later. The assumed $V(t,x_t,\xi,\gamma)$ is some smooth function.

Comment:

The stochastic integral might be defined by a stochastic integration in the mean-square sense of Ito or Stratonovich. The Ito integral is much easier for computation of expectation of the Ito integral

than the Stratonovich integral, and it has other nice mathematical properties not possessed by the Stratonovich integral. On the other hand, the Ito stochastic differential rule as given by the following theorem states the conditions where by a certain random process is permitted a stochastic differential [64].

Theorem 3.1

Let the function $f(t,x_t,\xi,\gamma)$ be a measurable smooth function which has partial derivatives f_t, f_{x_t}, f_ξ, f_γ, $f_{x_t x_t}$, $f_{\xi\xi}$, $f_{x_t\gamma}$, $f_{\xi\gamma}$, $f_{\gamma\gamma}$. The Ito formula is then given by

$$\begin{aligned} d\, f(t,x_t,\xi,\gamma) = {} & f_t(t,x_t,\xi,\gamma)dt + f_{x_t}(t,x_t,\xi,\gamma)dx_t \\ & + f_\xi(t,x_t,\xi,\gamma)d\xi + f_\gamma(t,x_t,\xi,\gamma)d\gamma + \\ & \frac{1}{2} f_{x_t x_t}(t,x_t,\xi,\gamma)\bar{G}\bar{G}^* dt \\ & + \frac{1}{2} f_{\xi\xi}(t,x_t,\xi,\gamma)\ K(t)\ K(t)^* dt, \end{aligned} \tag{3-13}$$

where $K(t) = \gamma F(t)^* (H(t)\ H(t)^*)^{-1}$.

Proof: Omitted (see [64]). ■.

These stochastic integrals suggest that the correct formula for $df(t,x_t,\xi,\gamma)$ is (3-13) where $\bar{G}$ and K stem from parameters of (3-7) and $\Gamma_t F(t)^* (H(t)H(t)^*)^{-1}$, respectively.

Using theorem 3.1, the differential form of the value function (3-12) is given by

$$\begin{aligned} dV(t,x_t,\xi,\gamma) = {} & x_t^* \frac{\partial\Lambda_1}{\partial t} x_t dt + \frac{\partial\Lambda_2}{\partial t} x_t\ dt + 2(\bar{A}x_t + \bar{B}u_t)^* \Lambda_1 x_t dt \\ & + (\bar{A}x_t + \bar{B}u_t)^* \Lambda_2 dt + \mathrm{tr}(\bar{G}\bar{G}^* \Lambda_1)dt \\ & + (F\xi)^* (\frac{\partial}{\partial\xi}\ (x_t^* \Lambda_1 x_t + \Lambda_2 x\ _t + \Lambda_3)\big|_{\xi=\hat{z}_t})dt \end{aligned}$$

(cont.)

$$+ (D(t)D(t)^* - \gamma F(t)^*(H(t)H(t)^*)^{-1}F(t)\gamma$$
$$+ C(t)\gamma + \gamma C(t^*))^* \cdot (\frac{\partial}{\partial\gamma}(x_t^* \Lambda_1 x_t + \tag{3-14}$$
$$\Lambda_2 x_t + \Lambda_3)|_{\gamma=\Gamma_t})dt$$
$$+ \frac{1}{2} \mathrm{tr}\,(KK^*(\frac{\partial^2}{\partial\xi\partial\xi^*}(x_t^*\Lambda_1 x_t + \Lambda_2 x_t + \Lambda_3)|_{\xi=\hat{z}_t}))dt$$
$$+ (2x_t^*\Lambda_1 + \Lambda_2)\bar{G}\, dw_t^1$$
$$+ (\frac{\partial}{\partial\xi}(x_t^*\Lambda_1 x_t + \Lambda_2 x_t + \Lambda_3)|_{\xi=\hat{z}_t})Kd\nu_t.$$

Taking the integral of both sides of (3-14), it follows that

$$V(T,x_T,\hat{z}_T,\Gamma_T) - V(0,x_0,\hat{z}_0,\Gamma_0) = \int_0^T (x_t^*(L(\Lambda_1) + 2\bar{A}^*\Lambda_1)x_t$$
$$+ 2(\bar{B}u_t)^*x_t + L(\Lambda_2) + \bar{A}^*\Lambda_2)x_t + \mathrm{tr}(\bar{G}\bar{G}^*)\Lambda_1$$
$$+ (\bar{B}u_t)^*\Lambda_2 + L(\Lambda_3))dt + \int_0^T ((2x_t^*\Lambda_1 + \Lambda_2)\bar{G}\, dw_t^1$$
$$+ \frac{\partial}{\partial\xi}(x_t^*\Lambda_1 x_t + \Lambda_2 x_t + \Lambda_3|_{\xi=\hat{z}_t})kd\nu_t), \tag{3-15}$$

where

$$L(.) = \frac{\partial}{\partial t}(\cdot) + ((F\hat{z}_t)^*\frac{\partial}{\partial\xi}(\cdot) + \frac{1}{2}\mathrm{tr}(KK^*\frac{\partial^2}{\partial\xi\partial\xi^*}(\cdot)))|_{\xi=\hat{z}_t}$$
$$+ (DD^* - \gamma F^*(HH^*)^{-1}F\gamma + C\gamma + \gamma C^*)^*\frac{\partial}{\partial\gamma}(\cdot)|_{\gamma=\Gamma_t},$$

and all arguments (t,ξ,γ) are omitted for brevity. The equation of (3-15) is given by

$$E(V(T,x_T,\hat{z}_T,\Gamma_T) - V(0,x_0,\hat{z}_0,\Gamma_0))$$
$$= E(\int_0^T x_t^*(L(\Lambda_1) + \bar{A}^*\Lambda_1 + \Lambda_1\bar{A})x_t$$
$$+ 2(\bar{B}u_t)^*x_t + (L(\Lambda_2) + \bar{A}^*\Lambda_2)x_t \tag{3-17}$$
$$+ \mathrm{tr}(\bar{G}\bar{G}^*)\Lambda_1 + (\bar{B}u_t)^*\Lambda_2 + L(\Lambda_3))dt.$$

Consider the formal application of Bellman's principle of optimality along with the differential formula which suggests that V should satisfy the stochastic Bellman equation.

Theorem 3.2

The optimal control in the approximation stochasitc system (3-7) is given by

$$u_t^o = -\bar{R}^{-1}(t,\hat{z}_t,\Gamma_t)\,\bar{B}^*(t,\hat{z}_t,\Gamma_t)(\Lambda_1(t,\hat{z}_t,\Gamma_t)x_t + \frac{1}{2}\Lambda_2(t,\hat{z}_t,\Gamma_t)), \tag{3-18}$$

where $\Lambda_1(t,\hat{z}_t,\Gamma_t)$ and $\Lambda_2(t,\hat{z}_t,\Gamma_t)$ satisfy the following Riccati-like equations:

$$\begin{aligned}\dot{\Lambda}_1 &= -\bar{A}^*\Lambda_1 - \Lambda_1\bar{A} + \bar{Q} - \Lambda_1\bar{B}\bar{R}^{-1}\bar{B}^*\Lambda_1 - (F\xi)^*\frac{\partial}{\partial\xi}\Lambda_1 - \frac{1}{2}\,\mathrm{tr}(KK^*\frac{\partial^2}{\partial\xi\partial\xi^*}\Lambda_2)\\ &\quad -(DD^* - \gamma F^*(HH^*)^{-1}F\gamma + C\gamma + \gamma C^*)^*\frac{\partial}{\partial\gamma}\Lambda_1, \qquad (3\text{-}19)\\ \dot{\Lambda}_2 &= -\Lambda_2\bar{A} - \Lambda_2\bar{B}\bar{R}^{-1}\bar{B}^*\Lambda_1 - (F\xi)^*\frac{\partial}{\partial\xi}\Lambda_1 - \frac{1}{2}\,\mathrm{tr}(KK^*\frac{\partial^2}{\partial\xi\partial\xi}\Lambda_2)\\ &\quad -(DD^*-\gamma F^*(HH^*)^{-1}F\gamma + C\gamma + \gamma C^*)^*\frac{\partial}{\partial\gamma}\Lambda_2.\end{aligned}$$

The arguments (t,ξ,γ) are omitted for brevity, and $\Lambda_1(T,\xi,\gamma) = 0$, and $\Lambda_2(T,\xi,\gamma) = 0$.

Proof:

Under the same consideration of theorem 2.4 in Chapter 2, the stochastic Bellman equation is

$$\begin{aligned}&x_t^*(L(\Lambda_1) + \bar{A}^*\Lambda_1 + \Lambda_1\bar{A})x_t + x_t^*\bar{Q}x_t + L(\Lambda_2)x_t + \Lambda_2\bar{A}x_t\\ &\quad + L(\Lambda_3) + \mathrm{tr}(\bar{G}\bar{G}^*\Lambda_1) + \min_u\,(u_t^*\bar{R}u_t + 2u_t^*\bar{B}^*\Lambda_1x_t\\ &\quad + u_t^*\bar{B}^*\Lambda_2) = 0. \qquad (3\text{-}20)\end{aligned}$$

Note that the last term of (3-20) only depends on u_t. Hence,

$$u_t^* \bar{R} u_t + 2u_t^* \bar{B}^* \Lambda_1 x_t + u_t^* \bar{B}^* \Lambda_2$$

$$= (u_t + \bar{R}^{-1}(\bar{B}^* \Lambda_1 x_t + \frac{1}{2} \bar{B}^* \Lambda_2)^* \bar{R}(u_t + \bar{R}^{-1}(\bar{B}^* \Lambda_1 x_t + \frac{1}{2} \bar{B}\Lambda_2))$$

$$- (\bar{R}^{-1}(\bar{B}^* \Lambda_1 x_t + \frac{1}{2} \bar{B}^* \Lambda_2)(\bar{R}^{-1}(\bar{B}^* \Lambda_1 x_t + \frac{1}{2} \bar{B}^* \Lambda_2)).$$

Thus, the minimum is achieved at

$$u_t^o = -\bar{R}^{-1} \bar{B}^* (\Lambda_1 x_t + \frac{1}{2} \Lambda_2). \tag{3-21}$$

(3-20) is substituted by (3-21), and then

$$x_t^* (L(\Lambda_1) + \bar{A}^* \Lambda_2 + \Lambda_1 A) x_t + x_t^* \bar{Q} x_t + \Lambda_2 \bar{A} x_t + L(\Lambda_2) x_t + L(\Lambda_3)$$

$$+ \operatorname{tr}(\bar{G}\bar{G}^* \Lambda_1) - x_t^* \Lambda_1 \bar{B}\bar{R}^{-1} \bar{B}^* \Lambda_1 x_t - \Lambda_2 \bar{B}\bar{R}^{-1} \bar{B}^* \Lambda_1 x_t = 0. \tag{3-22}$$

Therefore, (3-22) is the solution as long as Λ_1, Λ_2, and Λ_3 satisfy

$$\frac{\partial \Lambda_1}{\partial t} + \bar{A}^* \Lambda_1 + \Lambda_1 \bar{A} + \bar{Q} - \Lambda_1 \bar{B}\bar{R}^{-1} \bar{B}^* \Lambda_1 + (F\hat{z}_t)^* \frac{\partial}{\partial \hat{z}_t} \Lambda_1$$

$$+ (DD^* - \Gamma_t^* (HH^*)^{-1} F\Gamma_t + C\Gamma_t + \Gamma_t C^*)^* \frac{\partial}{\partial \Gamma_t} \Lambda_1$$

$$- \frac{1}{2} \operatorname{tr}(KK^* \frac{\partial^2}{\partial \Gamma_t \partial \Gamma_t^*} \Lambda_1) = 0,$$

$$\frac{\partial \Lambda_2}{\partial t} + \Lambda_2 \bar{A} - \Lambda_2 \bar{B}\bar{R}^{-1} \bar{B}^* \Lambda_1 + (F\hat{z}_t)^* \frac{\partial}{\partial \hat{z}_t} \Lambda_2$$

$$+ (DD^* - \Gamma_t F^* (HH^*)^{-1} F\Gamma_t + C\Gamma_t + \Gamma_t C^*)^* \frac{\partial}{\partial \Gamma_t} \Lambda_2$$

$$- \frac{1}{2} \operatorname{tr}(KK^* \frac{\partial^2}{\partial \Gamma_t \partial \Gamma_t^*} \Lambda_2) = 0,$$

$$\frac{\partial \Lambda_3}{\partial t} + (F\hat{z}_t)^* \frac{\partial}{\partial \hat{z}_t} \Lambda_3 + (DD^* - \Gamma_t F^* (HH^*)^{-1} F\Gamma_t + C\Gamma_t$$

$$+ \Gamma_t C^*)^* \frac{\partial}{\partial \Gamma_t} \Lambda_3 - \frac{1}{2} \operatorname{tr}(KK^* \frac{\partial^2}{\partial \hat{z}_t \partial \hat{z}_t^*} \Lambda_3) + \operatorname{tr}(\bar{G}\bar{G}^* \Lambda_1) = 0.$$

$$\Lambda_1(T, \hat{z}_T, \Gamma_T) = 0, \quad \Lambda_2(T, \hat{z}_T, \Gamma_T) = 0, \quad \Lambda_3(T, \hat{z}_T, \Gamma_T) = 0. \quad \blacksquare$$

The above results for BHJ equation have the following inequality:

$$\hat{J}(u_t^{o}) \leq \hat{J}(u_t)\ , \quad u_t \varepsilon\ \mathcal{U}.$$

The control u_t^{o} defined by (3-18) is admissible since the stochastic equation (3-7) has a unique strong[+] solution.

Now take into account the stochastic system (3-1) with random coefficients. The different approximation method has also been modified by the proper model using the mean estimate of (3-2),

$$dx_t \cong A(t,\hat{z}_t)x_t dt + B(t,\hat{z}_t)u_t dt + G(t,\hat{z}_t)dw_t^{1}, \qquad (3\text{-}23)$$
$$x(0) = x_0.$$

Approximation model (3-23) is simpler than (3-7), the first approximation to stochastic system equation (3-1). If $A(t,z_t)$ does not have the form of $A(t)z_t$ in (3-7) and (3-23),

$$E[A(t,z_t)\,|\,\mathcal{Y}_t,\ 0 \leq t \leq T] \neq A(t,\hat{z}_t).$$

Therefore, the equations (3-7) and (3-23) are different approximation models, in general.

Consider the problem of optimal control of stochastic model (3-23) with the cost function (3-4). In this case the problem has the modified cost function which is the same as (3-9). Again, define the following value function corresponding to (3-23) as

$$V'(t,x_t,\xi) = x_t^{*}\ \Lambda_1'(t,\xi)x_t + \Lambda_2'(t,\xi)x_t + \Lambda_3'(t,\xi),$$

where Λ_1', Λ_2', and Λ_3' are symmetric matrices which satisfy certain nonlinear partial differential quations.

[+]note that strong and weak solutions are discussed in [64].

Theorem 3.3

Let Λ_1' and Λ_2' be the bounded symmetric solution of the following Riccati-like equation:

$$\dot{\Lambda}_1' = -A^*\Lambda_1' - \Lambda_1'A + \bar{Q} - \Lambda_1'B\bar{R}^{-1}B^*\Lambda_1' - (F\xi)^*\frac{\partial}{\partial\xi}\Lambda_1'$$
$$-\frac{1}{2}\,\mathrm{tr}(KK^*\frac{\partial^2}{\partial\xi\partial\xi^*}\Lambda_1'),$$

$$\dot{\Lambda}_2' = -\Lambda_2'A - \Lambda_2'B\bar{R}^{-1}B^*\Lambda_1' - (F\xi)^*\frac{\partial}{\partial\xi}\Lambda_2' - \frac{1}{2}\,\mathrm{tr}(KK^*\frac{\partial^2}{\partial\xi\partial\xi^*}\Lambda_2'),$$

where the arguments $(t,\xi) \in [0,T] \times R^k$ are omitted for brevity, and $\Lambda_1'(T,\xi) = 0$, $\Lambda_2'(T,\xi) = 0$, for $\xi \in R^k$. Then, the optimal control of (3-23) under proper assumptions is given by

$$u_t^o = -\bar{R}^{-1}(t,\hat{z}_t)B^*(t,\hat{z}_t)(\Lambda_1'(t,\hat{z}_t)x_t + \frac{1}{2}\Lambda_2'(t,\hat{z}_t)). \qquad (3\text{-}24)$$

Proof:

The proof is similar to theorem 3.2. ∎

Remark: The optimal controls of (3-7) and (3-23) are suboptimal for (3-1) because the class of stochastic models of (3-7) and (3-23) are approximation models of (3-1) using $E[z_t|Y_t,\ 0 \le t \le T]$ and $E[(z_t-\hat{z}_t)^2|Y_t,\ 0 \le t \le T]$.

3.2 Suboptimal Control of a Class of Stochastic System with Unobservable Random Parameters

In general, uncertain unobservable stochastic processes in (3-2) have nonlinear observation equations; then, the optimal mean-square estimate of unobservable states is the solution of the

certain infinite-dimensional equations. Hence, it is necessary to model an implementable approximation and evaluate its performance in (3-1). Let x_t, $t \in [0,T]$, satisfy the following Ito-type stochastic equation:

$$dx_t = A(t,z_t)x_t dt + B(t,z_t)u_t dt + G(t,z_t)dw_t^1 , \qquad (3\text{-}25)$$

where z_t is an unobservable stochastic process satisfying

$$dz_t = C(t,z_t)dt + D(t,z_t)dw_t^2. \qquad (3\text{-}26)$$

Assume that x_t and y_t are observed, where y_t satisfies

$$dy_t = F(t,z_t)dt + H(t)dw_t^2. \qquad (3\text{-}27)$$

Processes x_t, z_t, y_t, w_t^1, w_t^2 are of dimensions n, m, ℓ, q_1, q_2, respectively, and all matrix functionals A, B, G, C, D, F are of appropriate dimensions. It is also assumed that w_t^1 and w_t^2 are mutually independent. Each of the measurable functionals $A(t,\Theta)$, $B(t,\Theta)$, $G(t,\Theta)$, $C(t,\Theta)$, $D(t,\Theta)$, $F(t,\Theta)$ is assumed to be non-anticipative. The p-dimensional stochastic process u_t, referred to here as a control, is assumed to be $\mathcal{H}_t$-measurable, where $\mathcal{H}_t$ is the σ-algebra generated by $\{x_s, y_s, 0 \leq s \leq t\}$.

The problem is to find the control u_t that minimizes the cost functional,

$$J(u) = E\int_0^T L(t,x_t,u_t)dt. \qquad (3\text{-}28)$$

(3-25), (3-26), (3-27) may be interpreted as a linear stochastic control system with random, partially-observable parameters. If the stochastic differential equations (3-25), (3-26), (3-27) are satisfied in theorem 2.1 in Chapter 2, there exists a unique solution of (3-25), (3-26), (3-27), respectively.

Let $\mathcal{E}_t[\cdot] \triangleq \mathcal{E}[\cdot|\mathcal{Y}_t]$ where $\mathcal{Y}_t$ is the σ-algebra generated by $\{y_s,\ 0 \leq s \leq t\}$. The following result is now needed:

<u>Theorem 3.4.</u> If f is a c_2-class function and satisfies

$$\int_0^T \mathcal{E}\left| f(z_t)(HH^*)^{-\frac{1}{2}} F(t,z_t)\right| dt < \infty,$$

then,

$$\mathcal{E}_t[f(z_t)] \triangleq \mathcal{E}(f(z_t)|\mathcal{Y}_t],$$

satisfies the following stochastic differential equation:

$$\begin{aligned} d\mathcal{E}_t[f(z_t)] = \mathcal{E}_t[\tilde{C}f(z_t)]dt + \{&\mathcal{E}_t[f(z_t)F(t,z_t)] \\ &- \mathcal{E}_t[f(z_t)]\mathcal{E}_t[F(t,z_t)] + H(t)\mathcal{E}_t[\tilde{D}]\} \\ &(H(t)H(t)^*)^{-\frac{1}{2}}(dy_t - \mathcal{E}_t[F(t,z_t)]dt), \end{aligned} \tag{3-29}$$

where

$$\tilde{C}f = \sum_{i=1}^{m} C(t,z_t)_i \left(\frac{\partial f}{\partial z_t}\right)_i + \frac{1}{2}\sum_{i,j=1}^{m} (DD^*)_{ij}\left(\frac{\partial^2 f}{\partial z_t \partial z_t^*}\right)_{ij}, \tag{3-30}$$

and

$$\tilde{D}_i = \sum_{i=1}^{m} D_{ji}\left(\frac{\partial f}{\partial z_t}\right)_j . \tag{3-31}$$

<u>Proof</u>: Proof can be found in [69]. Here, if $f(z_t) = z_t$, (3-29) is the conditional first movement, and if $f(z_t) = z_t z_t^*$, (3-29) is the conditional second moment. ■

Assume now that m = 1 (z_t is a scalar process) and that $f(z_t) = z_t^N$, where N is a positive integer number.

Then, in general, the stochastic differential equation (3-25), (3-26), (3-27) and performance index (3-28) do not form a closed system as the equation for $dE_t[z_t^N]$ involves the next higher conditional moment. This means that solution to an infinite dimensional set of differential equations may be required in order to obtain conditional moments of z_t. However, further assumptions are made here so that either of the following situations occur:

1) The above equations form a finite-dimensional set, i.e., there exist $K > 1$ such that for all $N > K$ $E_t(z_t^N) = \text{function}\ (E_t(z_t), E_t(z_t^2), \ldots, E_t(z_t^K))$; or

2) an approximation technique is used to obtain a finite-dimensional set of equations for the Kth-conditional moment of z_t.

The information available to the controller allows for a better mean-square estimate of z_t which is given by $\hat{z}_t$ since x_t contains information about z_t as well as y_t. Here, it is assumed that the estimator $\hat{z}_t$ is suboptimal in the sense that only the observation y_s, $s \in [0,T]$, is used to construct the estimate. These assumptions decouple the control problem and estimation of z_t problem.

In order to solve the suboptimal minimization problem stated by (3-25) and (3-28), the following approximation of (3-25) is used:

$$dx_t \cong \bar{A}_t\, x_t dt + \bar{B}_t u_t dt + \bar{G}_t dw_t^1, \tag{3-32}$$

where

$$\bar{A}_t = E[A(t,z_t)|\mathcal{Y}_t],$$
$$\bar{B}_t = E[B(t,z_t)|\mathcal{Y}_t], \qquad (3\text{-}33)$$
$$\bar{G}_t = E[G(t,z_t)|\mathcal{Y}_t].$$

The above assumption transforms the partially-observable stochastic problem into a completely observable one. The tools for solving the above approximate control problem are still limited. Here, one may refer to [16], where a similar structure of controlled diffusion process is discussed for quadratic form of the cost function L. Note that in this case, a different approximation is used. It is assumed that $\bar{A}_t$, $\bar{B}_t$, $\bar{G}_t$ can be approximated or described exactly by a measurable function of the conditional moments $E_t[z_t^N]$, N = 1, 2, ..., K, ... With the assumptions made previously about finite dimensionality of the suboptimal filter for the estimator of z_t, it follows that

$$\bar{A}_t \cong \tilde{A}(t,\xi_t),$$
$$\bar{B}_t \cong \tilde{B}(t,\xi_t), \qquad (3\text{-}34)$$
$$\bar{G}_t \cong \tilde{G}(t,\xi_t),$$

where

$$\xi_t = (E_t(z_t), E_t(z_t^2), \ldots, E_t(z_t^K)), \qquad (3\text{-}35)$$

satisfies the set of equations of the form

$$d\xi_t = S(t,\xi_t)dt + P(t,\xi_t)d\nu_t. \qquad (3\text{-}36)$$

Now, the approximate, completely observable version of (3-25), (3-26), (3-27) takes the form of

$$dx_t = \tilde{A}(t,\xi_t)x_t dt + \tilde{B}(t,\xi_t)u_t dt + \tilde{G}(t,\xi_t)dw_t^{\,1}, \qquad (3\text{-}37)$$

$$d\xi_t = S(t,\xi_t)dt + P(t,\xi_t)d\nu_t,$$

where ν_t is a Wiener process independent of $w_t^{\,1}$ because of the independence of $w_t^{\,1}$ and $w_t^{\,2}$.

If the cost function J(u) in (3-28) is of a quadratic form, i.e.,

$$J(u) = E[\int_0^T (x_t^{\,*} Q(t) x_t + u_t^{\,*} R(t) u_t)dt + x_T^{\,*} M\, x_T]. \qquad (3\text{-}38)$$

The following results apply (29].

Let the following assumptions be satisfied for all $t \in [0,T]$, $\eta \in R^k$:

1) $\| \tilde{A}(t,\eta)\| + \| \tilde{B}(t,\eta)\| + \| \tilde{G}(t,\eta)\| \leq k \leq \infty$,

   where k is a finite positive contant;

2) $S(t,\eta)$, $P(t,\eta)$ are such that a unique strong solution of (3-46) exists and

   $$\text{Prob. } (\int_0^T \| \xi_t\|^2 dt < \infty) = 1;$$

3) Q(t) and M are non-negative definite, and R(t) is uniformly positive definite (i.e., its inverse is uniformly bounded), and

4) u_t satisfies

   $$\int_0^T E(\|u_t^{\,2}\|)dt \leq \infty.$$

Theorem 3.5 Under the above assumptions, if there exists a bounded solution $V(t,\eta)$ to the Cauchy problem,

$$LV + \tilde{A}^* V + V\tilde{A} + Q - V\tilde{B}\tilde{R}^{-1}\tilde{B}^* V = 0,$$

where

$$V(T,\eta) = M,$$

$$LV = \frac{\partial V}{\partial t} + (S^* \frac{\partial}{\partial \eta})V + \frac{1}{2}\,\mathrm{tr}(pp^* \frac{\partial^2}{\partial \eta \partial \eta^*})V,$$

and the arguments $(t,\eta) \in [0,T] \times R^k$ are omitted for brevity. Then, the optimal control exists, and it is given by

$$u_t^o = -R^{-1}(t)\; B^*(t,\xi_t)\; V(t,\xi_t)x_t.$$

Proof: Proof is the same as theorem 3.2. The above Cauchy problem has a solution if all of the coefficients of the Cauchy equation are Hölder continuous. It can then be shown that V is nonnegative and uniformly bounded on $[0,T] \times R^k$ [29]. ■

Remark: The measurement data available to the feedback controls permits for a conditional estimate of z_t which is given by $\mathcal{E}(z_t|\mathcal{Y}_t,\ 0 \leq t \leq T)$. The state x_t also contains the measurement information about z_t as well as y_t.

Consider the problem of estimating the unobservable state z_t, $t \in [0,T]$, on the basis of results of the observation x_t and y_t with the following stochastic system of equations:

$$dx_t = A(t,z_t)x_t dt + B(t)u_t dt + G(t)dw_t^1, \tag{3-47}$$

$$dz_t = C(t)z_t dt + D(t)dw_t^2, \tag{3-48}$$

$$dy_t = F(t)z_t dt + H(t)dw_t^3, \tag{3-49}$$

$$x(0) = x_0,\ z(0) = z_0,\ y(0) = y_0,$$

where z_t cannot be observed directly. Let $E(\alpha|Z_t,\ 0 \le t \le T)$ be $E_t(\alpha)$ and $\hat{\alpha}$. Under the appropriate assumptions, the conditional mean $E(z_t|Z_t,\ 0 \le t \le T)$ of the given σ-algebra Z_t is given by

$$d\hat{z}_t = C(t)\hat{z}_t dt + \left(E_t\left(z_t\begin{bmatrix} A(t,z_t)x_t \\ F(t)z_t \end{bmatrix}^*\right) - E_t(z_t)E_t\begin{bmatrix} A(t,z_t)x_t \\ F(t)z_t \end{bmatrix}^*\right)$$

$$\begin{bmatrix} 1/G(t) & 0 \\ 0 & 1/H(t) \end{bmatrix}\begin{bmatrix} d\nu_t^1 \\ d\nu_t^2 \end{bmatrix}, \tag{3-50}$$

where

$$\begin{bmatrix} d\nu_t^1 \\ d\nu_t^2 \end{bmatrix} = \begin{bmatrix} 1/G(t) & 0 \\ 0 & 1/H(t) \end{bmatrix}\left(\begin{bmatrix} dx_t \\ dy_t \end{bmatrix} - \begin{bmatrix} E_t(A(t,z_t))x_t dt + B(t)u_t dt \\ F(t)\hat{z}_t dt \end{bmatrix}\right) \tag{3-51}$$

is an innovation process with respect to Z_t.

The conditional variance equation $E((z_t-\hat{z}_t)(z_t-\hat{z}_t)^*|Z_t)$ becomes

$$d\Gamma_t = 2\{\,E_t[z_t z_t^*]C(t) - \hat{z}_t\hat{z}_t^*C(t)^*\}\,dt + D(t)D^*(t)dt$$

$$-\left\{E_t[z_t\begin{bmatrix} A(t,z_t)x_t \\ F(H)z_t \end{bmatrix}^*] - \hat{z}_t\begin{bmatrix} E_t\ A(t,z_t)x_t \\ F(t)\hat{z}_t \end{bmatrix}^*\right\}$$

$$\cdot\begin{bmatrix} 1/G(t)G^*(t) & 0 \\ 0 & 1/H(t)H^*(t) \end{bmatrix}\left\{E_t\begin{bmatrix} A(t,z_t)x_t \\ F(t)z_t \end{bmatrix}z_t^*\right.$$

$$- \begin{bmatrix} \mathcal{E}_t \, A(t,z_t)x_t \\ F(t)\hat{z}_t \end{bmatrix} \hat{z}_t^{*} \} dt + \{ \mathcal{E}_t \, z_t z_t^{*} \begin{bmatrix} A(t,z_t)x_t \\ F(t)z_t \end{bmatrix}]$$

$$- \mathcal{E}_t [z_t z_t^{*}] \begin{bmatrix} \mathcal{E}_t \, A(t,z_t)x_t \\ F(t)\hat{z}_t \end{bmatrix} - \hat{z}_t \mathcal{E}_t [z_t^{*} \begin{bmatrix} A(t,z_t)x_t \\ F(t)z_t \end{bmatrix}]$$

$$- \hat{z}_t^{*} \mathcal{E}_t [z_t \begin{bmatrix} A(t,z_t)x_t \\ F(t)z_t \end{bmatrix}] + 2 \hat{z}_t \hat{z}_t^{*} \begin{bmatrix} \mathcal{E}_t (A(t,z_t)x_t \\ F(t)\hat{z}_t \end{bmatrix} \}^{*}$$

$$\begin{bmatrix} 1/G(t) & 0 \\ 0 & 1/H(t) \end{bmatrix} \begin{bmatrix} d\nu_t^{1} \\ d\nu_t^{2} \end{bmatrix} . \qquad (3\text{-}52)$$

The conditional mean and variance given observations $\{x_s, y_s, s \in [0,T]\}$ have infinite dimension in (3-50) and (3-52), respectively. If there exist $\mathcal{E}(z_t | Z_t, 0 \le t \le T)$ and $\mathcal{E}((z_t - \hat{z}_t)^2 | Z_t, 0 \le t \le T)$, the state x_t is the solution of the following stochastic equation:

$$dx_t = \mathcal{E}_t(A(t,z_t))x_t dt + B(t)u_t dt + G(t)d\nu_t^{1}, \qquad (3\text{-}53)$$

where $d\,\nu_t^{1}$ is the same as in (3-51). If there exists the conditional density function $p(t,z_t,\hat{z}_t,\Gamma_t)$ corresponding to (3-50) and (3-52), then, (3-53) may be replaced by the following stochastic equation,

$$dx_t = \underset{\sim}{A}(t,\hat{z}_t,\Gamma_t)x_t dt + B(t)u_t dt + G(t)d\nu_t^{1}, \qquad (3\text{-}54)$$

where

$$\underset{\sim}{A}(t,\hat{z}_t,\Gamma_t) = \int_{-\infty}^{\infty} A(t,z_t)p(t,z_t,\hat{z}_t,\Gamma_t)dz_t = \mathcal{E}[A(t,z_t)|Z_t]. \qquad (3\text{-}55)$$

Consequently the equation (3-54) has been reformulated by observable processes (3-51) and (3-52).

Comment: The conditional estimate $E[x_t|Z_t, o \leq t \leq T]$ is the same as x_t because x_t is Z_t measurable.

The innovation process in (3-50) depends on the control variable u_t, and thus, the separation principle could not be applied to verify the optimal control in (3-54). If applied to the stochastic linear controller, this will show that a lower cost cannot be obtained with nonlinear controls. The optimal control cannot be found using any well-known methods. For practical applications the most important results desired are the necessary conditions of optimality that can be used for synthesis of optimal feedback control laws. The answer to this question is yet to be resolved.

4. SIMULATION OF THE STOCHASTIC CONTROL PROCESSES WITH RANDOM COEFFICIENTS

In previous chapters the problem of optimal control of stochastic control processes with random coefficients has been presented. To evaluate the performance of the optimal control for each stochastic model, it is necessary to synthesize the control law, the state equation and the solution of a nonlinear partial differential equation. A discretization technique is used to calculate the state, the Wiener process and the optimal control for simulation.

A pseudo-Wiener process is used for the generation of the Wiener process; the first method is introduced by a Bernoulli time series [70] using pseudo-uniform random numbers between 0 and 1; the second uses pseudo-Gaussian random numbers $N(0,1)$.

The solution of the nonlinear partial differential equation to the Cauchy problem of equation (2-18) uses a semi-discrete system of nonlinear ordinary differential equations. The semi-discrete system equations can be solved by integration of the ordinary differential equations [71,72,73].

A simple one-dimensional stochastic system has been presented by the state and optimal control to the value function $V(t,x_t,z_t) = x_t^* \Lambda_1(t,z_t)x_t + \Lambda_2(t,z_t)$. A practical application of the theoretical extension in Chapters 2 and 3 is presented here by the landing problem of the longitudinal motion of an aircraft in a gusty wind.

A landing aircraft may be described approximately by a second-order differential equation with random parameters [44]. Certain

information data are derived by conditional estimation using the given observations. On the basis of observations, the conditional estimate is applied to derive the appropriate stochastic models. This aircraft-landing model is used to illustrate the design procedure of optimal control under the worst weather situation. The random parameter is assumed to be the result of a gusty wind. The models of wind, based on the Dryden model for turbulence and its aerodynamic effects, are used in conjunction with optimal-control design.

If the system is subject to both parameter uncertainty and noise disturbances, control of the dynamic system is treated by stochastic control theory. The problem of controlling a longitudinal motion of an aircraft in wind gust is very similar to the above problem [34]. The statistical properties of uncertain quantities, which are Lebesque-measurable functions whose values may range with proper boundaries, are assumed to be known, and the stochastic model of longitudinal motion of aircraft in wind gust is approximated by the second-order stochastic differential equation. It is also found that the simulation results of the motion of aircraft in wind gusts give a reasonable degree of approximation. Using the theoretical results of Chapter 2 and 3, the angle of attack, the orientation rate of aircraft, the active elevator control angle, and the active aileron control angle in a gusty wind are determined.

4.1 The Generation of Wiener Process

Let (Ω', F', P') be a probability space, W_t, $t \geq 0$, be a Wiener process, and F'_t be a nondecreasing family of sub-σ-algebra of F'. Then, the Wiener process has the following properties:

i) $E(W_t - W_s \,|F'_s) = 0$, $E((W_t - W_s)^2 |F'_s) = t-s$, $t \geq s$;

ii) $W_{t_2} - W_{t_1}$ and $W_{t_4} - W_{t_3}$ are independent for nonoverlapping intervals $[t_2, t_1]$ and $[t_4 - t_3]$;

iii) the trajectories W_t are continuous with probability 1;

iv) $W(0) = W_0 = 0$.

The construction of the Wiener process is most useful to study the stochastic model. For convenience, the Wiener process is approximated with the digital computer by a pseudo-Wiener process according to the properties of the Wiener process. A continuous stochastic integral for computation can be approximated by the following discrete Wiener process:

$$\int_0^t g(t)dW_t = \sum_{i=0}^{n-1} g(t_i)(W_{t_{i+1}} - W_{t_i}), \tag{4-1}$$

where $g(t)$ is some suitable function. The discrete representation of (4-1) should be kept in an infinite sequence of independent Gaussian variables on Ω'.

Consider first, the construction by which a Bernoulli time series can be made to approximate the Wiener process as follows: with fixed $\Delta > 0$, $\Delta \varepsilon [0,T]$, and the interpolation of certain

random sums,

$$W_t(\Delta) = h(\Delta)\Sigma\, z_i \ , \quad i \le t/_\Delta , \quad 0 \le t \le T, \tag{4-2}$$

where z_i is the Bernoulli variable which has a mean 0 and a magnitude 1. $W_t(\Delta)$ converges to the Wiener process under proper choice of the scalar $h(\Delta)$ [53]. If $\{z_i\}$ is a sequence of independent random variables such that

$$P[z_i=1] = P[z_i = -1] = \frac{1}{2}, \tag{4-3}$$

and the graph of $W_t(\Delta)$ is a scalar random walk. The scalar $h(\Delta)$ in (4-2) is selected so that regardless of the value Δ, the variance of $W_t(\Delta) = t$. Thus determine $h(\Delta)$ as follows:

$$\begin{aligned} \mathrm{Var}[W_t(\Delta)] &= t = h(\Delta)^2 \sum_{i \le t/\Delta} \mathrm{Var}[z_i], \\ &= [h(\Delta)]^2[\text{integer part of } (t/\Delta)], \\ &\cong \frac{t}{\Delta}\, [h(\Delta)]^2. \end{aligned} \tag{4-4}$$

Therefore $h(\Delta) = \sqrt{\Delta}$, and Laplace's theorem states

$$\sqrt{n} \ \left(\sum_{i=1}^{n} z_i\right) \xrightarrow{n \to \infty} N(0,t). \tag{4-5}$$

If n is t/Δ, then $h(\Delta) = \sqrt{t/n}$ and letting $\Delta \to 0$,

$$W_t(\Delta) = h(\Delta) \sum_{i=1}^{n} z_i = \sqrt{t/n} \ \left(\sum_{i=1}^{n} z_i\right). \tag{4-6}$$

Since z_i are independent, for $t_1 < t_2 < \ldots < t_n = T$, W_{t_1}, $W_{t_2} - W_{t_1}$, ..., $W_{t_n} - W_{t_{n-1}}$ are statistically independent and the stochastic

process $W_t(\Delta)$ converges, as $\Delta \to 0$, to the Wiener process W_t.

If there exist independent uniform random variables $r_1, r_2, \ldots, r_{nxm}$ between 0 and 1, define the following random variables as

$$z_i = \begin{cases} 1 & 0 \leq r_i \leq 0.5 \\ -1 & 0.5 < r_i \leq 1.0, \; i = 1,2,\ldots, n \times m. \end{cases}$$

Hence, $p(x_i=1)$ and $p(x_i=-1)$ are 1/2. The approximate Wiener process is given by

$$W_{t_j} = \sqrt{\frac{t_j}{nxm}} \left(\sum_{i=1}^{nxm} z_i \right), \quad j = 1,2,\ldots,m, \; i = 1,2,\ldots,n \times m, \tag{4-7}$$

where $n \times m = t_j/\Delta$ is total number of random variables z_i between t_j and 0, and Δ is the small time interval $t_j - t_{j-1}$. Equation (4-7) is equivalent to

$$W_{t_j} = \sqrt{\Delta} \left(\sum_{i=1}^{n(m-1)} z_i \right) + W_{t_{j-1}} \quad . \tag{4-8}$$

Figure 4.1 shows the normalization curve of (4-8) for $t \in [0,1]$, $\Delta = 0.001$, $n = 1000$, $m = 10$. 10,000 total uniform pseudo-random variables are used to generate 1,000 discrete Wiener processes.

Let W_j be a normal random sequence. In this case, $(W_{t_j} - W_{t_{j-1}})$ has the mean 0 and variance $(t_j - t_{j-1})$. Hence, the standard Brownian motion is always calculated by simply taking the linear function of $(W_{t_j} - W_{t_{j-1}})$. If $(W_{t_j} - W_{t_{j-1}}) = W_j$ has the distribution $N(0,t_j-t_{j-1})$), $W_j' = W_j/(t_j-t_{j-1})$ has the distribution $N(0,1)$. Table 4.1 shows the distribution of W_j' for 1000 normal random variables in the range of $-3.5 < W_j' < 3.5$ with the distribution of W_j' being

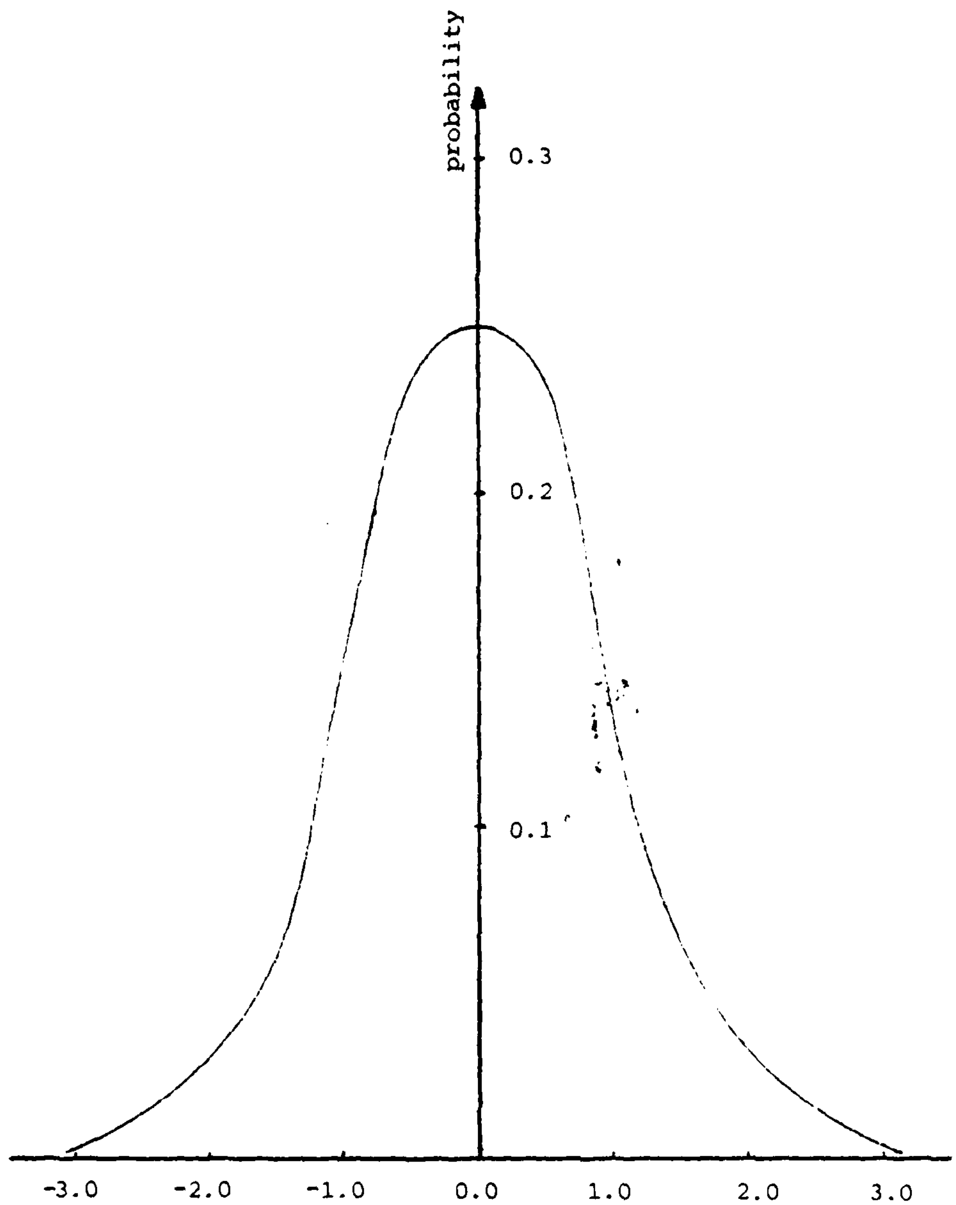

Figure 4.1. Normalization curve of the pseudo-Wiener process

TABLE 4.1 THE DISTRIBUTION OF THE NORMALIZED PSEUDO-WIENER PROCESS FOR 1000 RANDOM NUMBERS

Range	Numbers	Probability of Each Range
$-3.5 \leq W_j' < -3.0$	1	0.001
$-3.0 \leq W_j' < -2.5$	10	0.010
$-2.5 \leq W_j' < -2.0$	0	0.000
$-2.0 \leq W_j' < -1.5$	41	0.041
$-1.5 \leq W_j' < -1.0$	102	0.102
$-1.0 \leq W_j' < -0.5$	204	0.204
$-0.5 \leq W_j' < 0$	0	0.000
$W_j' = 0$	251	0.251
$0.0 < W_j' \leq 0.5$	0	0.000
$0.5 < W_j' \leq 1.0$	217	0.217
$1.0 < W_j' \leq 1.5$	126	0.126
$1.5 < W_j' \leq 2.0$	37	0.037
$2.0 < W_j' \leq 2.5$	0	0.000
$2.5 < W_j' \leq 3.0$	10	0.010
$3.0 < W_j' \leq 3.5$	1	0.001

$$P(-3 < W_j' < 3) = 0.998 \cong 1.0.$$

As illustrated by Figure 4.1 the density and the results of Table 4.1, it is a good approximation for 1000 discrete Wiener process.

These random variables are checked for the student t-distribution which (n-1) degree of freedom. The normalized Wiener increments W_j' in Table 4.2 is given by the t-test. All t values in Table 4.2 are less than the critical values, and therefore, the results satisfy the t-distribution. Hence, the pseudo-Wiener process is assumed to be equivalent to the Wiener process.

TABLE 4.2 THE RESULTS OF THE STUDENT t-DISTRIBUTION

Numbers of Normal Random Variables Used	Critical Values	The Given t Value
100	3.389	1.3493
200	3.389	1.5315
500	3.310	1.7067
1,000	3.291	1.8862

Apply the above Wiener-process simulation results to the simple first-order stochastic system

$$dx_t = A(t)x_t dt + G(t)dW_t, \tag{4-9}$$

where the mean m_t and variance S_t of x_t are

$$\dot{m}_t = A(t)m_t \quad , \quad m(0) = m_0,$$
$$\dot{S}_t = 2A(t)S_t + G(t)^2 \quad , \quad S(0) = S_0.$$

The discretization of (4-9) is

$$x_{t_i} = (1 + \delta A)x_{t_{i-1}} + G(t_{i-1})[W_{t_i} - W_{t_{i-1}}],$$

$$i = 0, 1, 2, \ldots, n - 1, \quad \delta = t_t - t_{t-1}, \quad n\delta = T.$$

Table 4.3 shows the t-test for A = -0.5 and -1.0, G = 1.0, x_0 = 1.0, S_0 = 0.0 with the other conditions being the same as before. Table 4.4 presents the Kolmogorou-Smirnov test results for A = -1.0 and x_o = 0.5. The other simulation conditions are the same. If the Kolmogorcv-Smirnov statistic exceeds the critical value, then the generated random variables should reject the hypothetical distribution. Test results are less than the critical values that have the significance level of 0.01 [74]. Formally it turns out that most of the entries in Table 4.4 passed the Kolmogorov-Smirnov test.

The different generation of the Wiener process W_j of Kolodziej [29] used pseudo-random Gaussian variables V_j which are $N(0,1)$ from the IMSL library called GGNML. Increments of the Wiener process W_t were approximated by the formula $dW_i(\Delta t) \cong \sqrt{\Delta t}\, V_i$.

Comment: Other generation methods of the Wiener process are presented by [75,76,77] using Walsh functions and Harr functions.

TABLE 4.3 THE RESULTS OF THE STUDENT t-DISTRIBUTION OF A STOCHASTIC SYSTEM FOR A = -1.0 AND -0.5

The Value of A	Number of Normal Random Variables Used	Critical Values	The given t Value
-1.0	100	3.389	0.9758
	200	3.389	1.3923
	500	3.310	1.2077
	1,000	3.291	1.7027
-0.5	100	3.389	0.6712
	200	3.389	0.8791
	500	3.310	0.4978
	1,000	3.291	0.6442

TABLE 4.4 THE TEST RESULTS OF KOLMOGOROV-SMIRNOV

Sample Size	Significance Level	Critical Value	Results
2	0.01	0.929	0.699
5	0.01	0.669	0.00009079
10	0.01	0.486	0.000
100	0.01	0.180	0.000

4.2 Numerical Solutions of Nonlinear Partial Differential Equations

The problem of solving (2-13) for given $\Lambda_1(T,z_T)$, and $\Lambda_2(T,z_T)$ is called the Cauchy problem. The solution is understood to be continuous in $R^n \times [0,T]$ and to have continuous derivatives $\frac{\partial\Lambda_1}{\partial t}$, $\frac{\partial\Lambda_2}{\partial t}$, $\frac{\partial\Lambda_1}{\partial\xi}$, $\frac{\partial\Lambda_2}{\partial\xi}$, $\frac{\partial^2\Lambda_1}{\partial\xi\partial\xi^*}$, $\frac{\partial^2\Lambda_2}{\partial\xi\partial\xi^*}$ in $R^n \times [0,T]$. The solution to problems in (2-13) may be transformed to initial-boundary value problems for t replaced by T-s, $t,s \in [0,T]$. Then, (2-13) becomes the classical Cauchy problem with the initial conditions instead of the terminal conditions.

Let

$$\frac{\partial\Lambda_k}{\partial t} = f_k\left(t,\xi,\Lambda_1,\Lambda_2,\ldots,\Lambda_{Np}, \frac{\partial\Lambda_1}{\partial\xi}, \frac{\partial\Lambda_2}{\partial\xi}, \ldots, \frac{\partial\Lambda_{NP}}{\partial\xi}, \frac{\partial^2\Lambda_1}{\partial\xi\partial\xi^*}, \frac{\partial^2\Lambda_2}{\partial\xi\partial\xi^*}, \ldots, \frac{\partial^2\Lambda_{NP}}{\partial\xi\partial\xi^*}\right), \tag{4-10}$$

$k = 1, 2, \ldots, NP,$

denote the coupled systems of partial differential equations with the initial conditions

$$\Lambda_1|_{s=0} = k_1, \quad \Lambda_2|_{s=0} = k_2, \ldots, \Lambda_{NP}|_{s=0} = k_{NP}, \tag{4-11}$$

where k_i is some constant. Kolodziej [29] proved that the partial differential equation (4-10) with (4-11) has a bounded unique solution.

The numerical solution of the nonlinear partial differential equation is complicated and is a highly problem-dependent process.

The semi-discrete system of nonlinear ordinary differential

equations is solved using one of the recently developed ordinary differential equations integrator [71,72]. The numerical method of lines [42] will be used for equation (4-10). Describe the finite difference approximations used by the computer simulation. Assume that a user has specified a time-independent spatial mesh which consists of a sequence of $NS(\geq 3)$ points in $[a,b]$ such that $a = \xi_1 < \xi_2 < \ldots < \xi_{NS} = b$. Define the mesh spacing as $\Delta_i = \xi_{i+1} - \xi_i$, for $i = 1,2,\ldots,NS-1$. Associate with this mesh the functions $\Lambda_{j,i}(s)$, $j = 1,2,\ldots,NP$ where j is the number of the partial differential equations, and $i = 1,2,\ldots,NS$. The value of the function $\Lambda_{j,i}(t)$ at any time t is meant to approximate the true solution value $\Lambda_j(s,\xi_j)$. To obtain an ordinary differential equation which will determine $\Lambda_{j,i}(s)$, evaluate the jth partial differential equation (4-10) at $\xi=\xi_i$ where $1 < i < NS$. For numerical solution, evaluation of Λ, Λ_ξ, and

$$\frac{\partial}{\partial \xi}\left(\frac{\partial \Lambda_j}{\partial \xi}\right), \qquad j = 1,2, \ldots , NP,$$

is necessary. Let the approximations $\Lambda_{j,i}(s)$ denote $\Lambda_{j,i}$. For $\xi=\xi_i$ in the jth partial differential equation, approximate

$$\Lambda_j(s,\xi) \cong \Lambda_{ji},$$

$$\frac{\partial \Lambda_j(s,\xi)}{\partial \xi} \cong \frac{\Lambda_{j(i+1)} - \Lambda_{j(i-1)}}{\Delta_j + \Delta_{i-1}},$$

$$\frac{\partial}{\partial \xi}\left(\frac{\partial \Lambda_j(s,\xi)}{\partial \xi}\right) \cong \frac{1}{(\xi_{i+1}+\xi_i)/2 - (\xi_{i-1}+\xi_i)/2} \cdot \left(\frac{\Lambda_{j(i+1)} - \Lambda_{ji}}{\Delta_i} - \frac{\Lambda_{ji} - \Lambda_{j(i-1)}}{\Delta_{i-1}}\right), \; j=1,2,\ldots, NP. \tag{4-12}$$

It remains to consider the points i = 1 and i = NS. For i = 1, consider the approximation at $\xi=\xi_1$

$$\Lambda_j(t,\xi) \cong \Lambda_{j1},$$

$$\frac{\partial\Lambda_j(t,\xi)}{\partial\xi} \cong \frac{\Lambda_{j2}-\Lambda_{j1}}{\Delta_1},$$

$$\frac{\partial}{\partial\xi}\frac{\partial\Lambda_j(t,\xi)}{\partial\xi} \cong \frac{(\Lambda_{j-3}-\Lambda_{j2})/\Delta_1 - (\Lambda_{j2}-\Lambda_{j1})/\Delta_1}{\Delta_1}. \tag{4-13}$$

For i = NS, its approximation is given by

$$\Lambda_j(t,\xi) \cong \Lambda_{jNS},$$

$$\frac{\partial\Lambda_j(t,\xi)}{\partial\xi} \cong \frac{\Lambda_{jNS} - \Lambda_{j(NS-1)}}{\Delta_{NS}}, \tag{4-14}$$

$$\frac{\partial}{\partial\xi}\frac{\partial\Lambda_j(t,\xi)}{\partial\xi} \cong \frac{(\Lambda_{jNS} - \Lambda_{j(NS-1)})/\Delta_{NS} - (\Lambda_{j(NS-1)} - \Lambda_{j(NS-2)})/\Delta_{NS}}{\Delta_{NS}}.$$

At this point, it should be clear that the finite difference approximations (4-12), (4-13), and (4-14) are substituted into (4-10). Now, the semi-discrete system of NP x NS approximate equation has the form

$$\frac{d\Lambda_{j1}}{dt} = F_{j1}(t,\Lambda_{j-1}, \Lambda_{j-2}, \Lambda_{j-3}),$$

$$\vdots$$

$$\frac{d\Lambda_{ji}}{dt} = F_{ji}(t,\Lambda_{j(i-1)}, \Lambda_{ji}, \Lambda_{j(i+1)}), \tag{4-15}$$

$$\vdots$$

$$\frac{d\Lambda_{jNS}}{dt} = F_{jNS}(t,\Lambda_{j(NS-2)}, \Lambda_{j(NS-1)}, \Lambda_{jNS}), \quad i=2,3,\ldots,(NS-1),$$

for j=1,2,...,NP, where $\Lambda_{ji} = (\Lambda_{1i}, \Lambda_{2i}, \ldots, \Lambda_{NPi})$. Since the F_{ji} no longer depends on only spatial derivatives, equation (4-15) is simply an approximating system of ordinary differential equations that are easily obtained from (4-11) with $\Lambda_{ji}\big|_{s=o} = k_j$ for i=1,2,..., NS and j=1,2,..., NP.

Most of the recently developed and currently available ordinary partial-differential-equation, integration routines [71,72,73] are designed to solve the initial-value problem for ordinary differential equations (4-15) where Λ, $\partial\Lambda/\partial S$, F are vector functions.

Example 4.1 Let $\Lambda(t,\xi)$, $\xi \in R^1$ satisfy

$$\frac{\partial\Lambda}{\partial t} + \frac{1}{2}\frac{\partial^2\Lambda}{\partial\xi^2} + 2\,A(t,\xi)\Lambda + 1.0 - \Lambda^2 = 0, \qquad (4\text{-}16)$$

and

$$\Lambda\,(T,\xi) = 0 \quad , \quad t \in [0,T]$$

where ξ_t is the solution of the following stochastic process

$$d\xi_t = dw_t \ , \quad \xi_0 = 0.$$

Here, w_t is normal Brownian motion. Then, ξ_t by itself is Gaussian. Let m and σ^2 be the mean and variance of ξ_t respectively. Choose $a = m - 3\sigma = -3\sigma$ and $b = m + 3\sigma = 3\sigma$. Hence, $a = -3$ and $b = +3$. Therefore,

$$\text{Prob.}(\xi \in [-3,3]) \approx 0.995 \cong 1.0.$$

The variable t is transformed by T - t, and then, (4-16) becomes

$$\frac{\partial\Lambda}{\partial t} = \frac{1}{2}\frac{\partial^2\Lambda}{\partial\xi^2} + 2A(t,\xi)\Lambda - 1.0 + \Lambda^2.$$

Let

$$\Delta_i = 0.1, \quad i = 1,2,\ldots, 61,$$

$$\Delta_t = 0.001, \quad t \in [0,1],$$

$$A = A_o + A_1 \tan^{-1}(\xi),$$

$$A_o = 3.75,$$

$$A_1 = 1.50.$$

Figure 4.2 shows the numerical solution of (4-16) for the five different values: a) $A_o = 3.75$, and $A_1 = 1.50$, b) $A_o = 3.2$, and $A_1 = 1.2$, c) $A_o = 2.5$, and $A_1 = 1.0$, d) $A_o = 2.0$, and $A_1 = 0.75$, e) $A_o = 1.5$, and $A_1 = 0.5$. The solution Λ is similar to the following Riccati equation:

$$\frac{\partial\Lambda'}{\partial t} = 2A_o\Lambda' - 1.0 + \Lambda'^2.$$

Example 4.2 Consider the following stochastic differential equation

$$dx_t = (3.75 + 1.5\tan^{-1} z_t)x_t dt + u_t dt, \quad x(0) = 1.0, \tag{4-17}$$

$$dz_t = dw_t, \tag{4-18}$$

where x_t is unobservable and w_t, $t \in [0,1]$ is a Wiener process. The solution to the optimal-control problem yields a control u_t^o that minimizes the criterion

$$J(u) = E[\int_0^1 (x_t^2 + u_t^2)dt].$$

According to the results discussed in [29], the stochastic control problem in (4-17) and (4-18) has the solution of the form

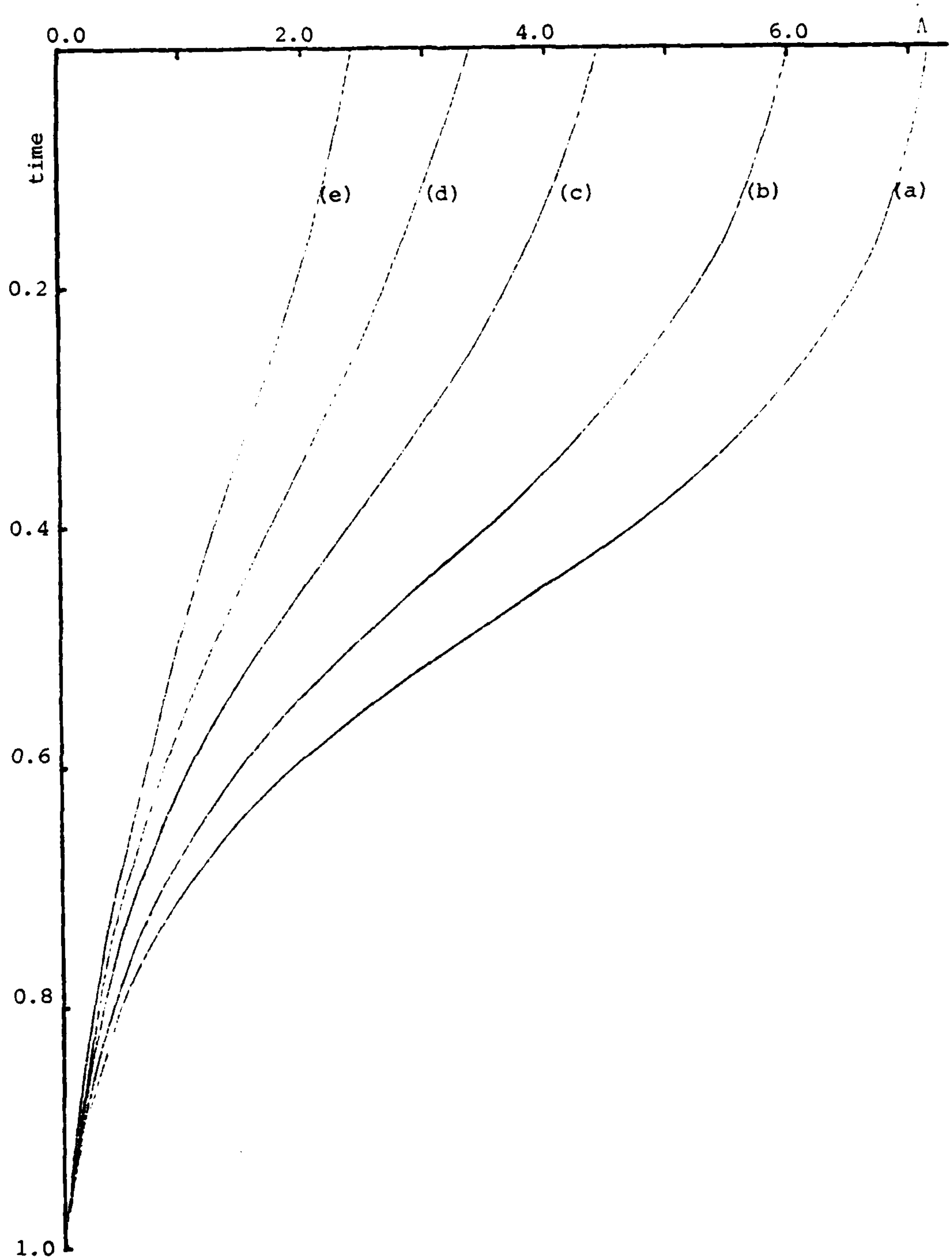

Figure 4.2 Solutions of the Riccati equation for (a) $A_o = 3.75$, (b) $A_o = 3.2$, $A_1 = 1.20$, (c) $A_o = 2.5$, $A_1 = 1.0$, (d) $A_o = 2.0$, $A_1 = 0.75$, (e) $A_o = 1.5$, $A_1 = 0.5$

$$u_t^{\,o} = - V(t,z_t)\hat{m}_t,$$

where

$$d\hat{m}_t = (3.75 + 1.5 \tan^{-1} z_t - V(t,z_t))\hat{m}_t\, dt, \quad \hat{m}_0 = x_0,$$

and $V(t,\xi)$, $\xi \in R$ satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial \xi^2} + 2\cdot(3.75 + 1.5 \tan^{-1} z_t)V + 1.0 - V^2 = 0,$$

$$V(1,\xi) = 0, \quad t \in [0,1].$$

Using the simulation results in example 4.2, Figure 4.3 shows that optimal control and suboptimal control obtained for (4.17) with $(3.75 + 1.5 \tan^{-1} z_t)$ replaced by $E[3.75 + 1.5 \tan^{-1} z_t]$. Figure 4.4 shows the sample paths according to optimal control and suboptimal control in (4-17).

4.3 Application to an Aircraft Landing Problem

Landing aircraft may be described approximately by a second-order differential equation with random coefficients [44].

$$\frac{d^2h(t)}{dt^2} = r(\sigma, v, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5)u_t, \tag{4-19}$$

where $h(t)$ is the altitude; u_t is the altitude control signal; r is a coefficient depending on air density σ, the flight velocity v, and aerodynamic coefficients δ_1, δ_2, δ_3, δ_4, δ_5. At the beginning of the landing process, the initial conditions are given by $h(0)$ and $\dot{h}(0)$. The flight velocity v is assumed to be constant throughout the landing process with finite inverval of landing time T. The coefficient r, which characterizes the objective of

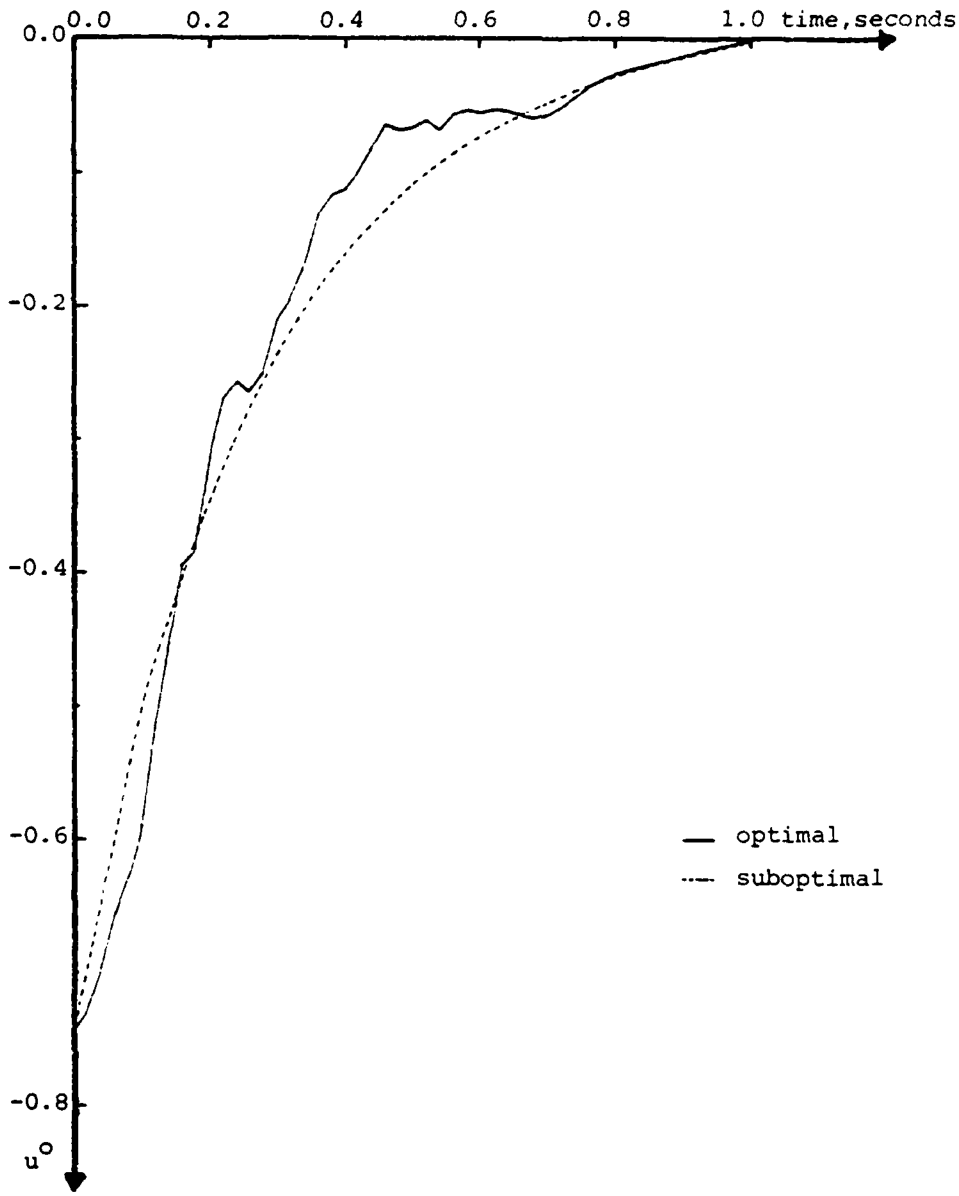

Figure 4.3 Optimal control law and suboptimal control law for example 4.2

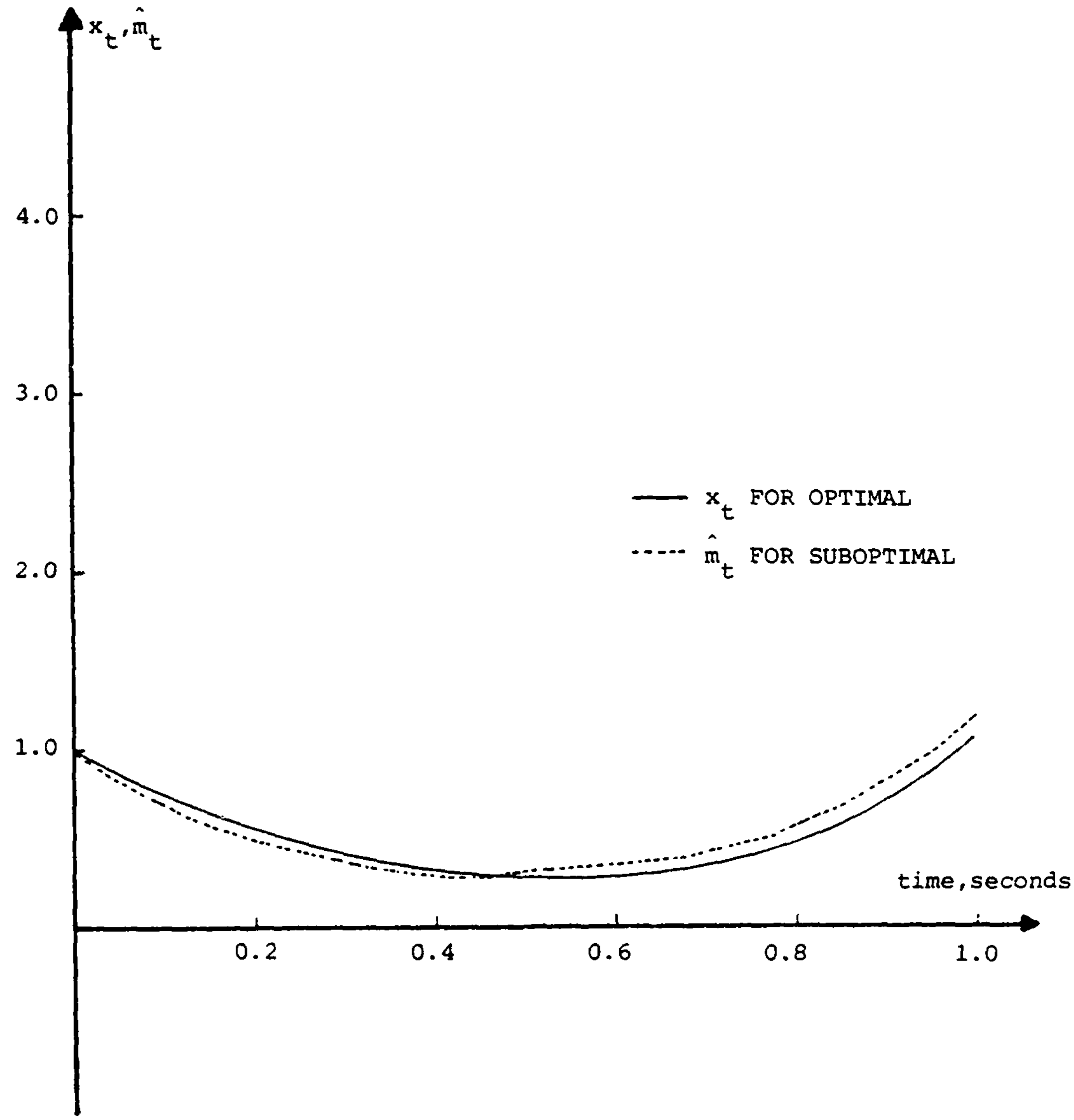

Figure 4.4 Realization of x_t under optimal and suboptimal for example 4.2

control, depends on many factors that can vary randomly at any moment in time. If the altitude has an additive noise process, then (4-19) may be modified as follows:

$$\frac{d^2h(t)}{dt^2} = r(t,z_t)u_t + N_t, \tag{4-20}$$

where N_t is a white noise process. Letting the state variable x_t represent the descent rate h(t), the stochastic model in the sense of Ito differential form may be described by

$$dx_t = r(t,z_t)u_t dt + g(t)dw_t^1, \; x(0) = x_0, \; x_t \in R^1. \tag{4-21}$$

the equation (4-21) is meaningful insofar as it's an integral equation [55]. It is assumed here that the random parameter r is dependent on a certain unobservable stochastic process,

$$dz_t = C(t,z_t)dt + D(t)dw_t^2, \quad z(0) = z_0, \quad z_t \in R^1,$$

and y_t is an observable process satisfying

$$dy_t = F(t,z_t)dt + H(t)dw_t^3.$$

Here w_t^1, w_t^2, w_t^3 are mutually independent Wiener processes.

The cost fuction, which depends on the rate of descent of the aircraft during the landing interval and the control system u_t, is given in the quadratic form

$$J(u) = E[\int_0^T u_t^2 dt + Mx_T^2]. \tag{4-22}$$

The above control system has the structure discussed in Chapter 3, and the corresponding results can be applied here. Assuming that

the first two conditional moments of z_t satisfactorily estimate z_t, the following approximation of (4-20) can be used:

$$dx_t = \tilde{r}(t,\hat{z}_t,\Gamma_t)u_t dt + g(t)dw_t^1, \tag{4-23}$$

where $\hat{z}_t$ is $E[z_t|Y_t]$ and $\Gamma_t = E[(z_t-\hat{z}_t)^2|Y_t]$.

One of the possible models for the wind gust velocity is suggested by Balakrishnan [11,42] such that

$$\dot{z}_t = -\alpha z_t + \beta n_t, \; \alpha > 0, \; \beta > 0, \tag{4-24}$$

and n_t is a white Gaussian noise with unit spectral density. Hence z_t is Gaussian, asymptotically stationary, with the spectral density function given by

$$P(\omega) = \frac{\beta^2}{4\pi^2\omega^2+\alpha^2}, \tag{4-25}$$

where $P(\omega)$ is called the spectral distribution of z_t. Equation (4-24) has its stochastic differential form as

$$dz_t = -\alpha z_t dt + \beta dw_t^2, \tag{4-26}$$

and the observation process is assumed to be of the form

$$dy_t = fz_t dt + dw_t^3. \tag{4-27}$$

Here w_t^1, w_t^2, w_t^3 are assumed to be mutually independent Wiener processes. Now the suboptimal filter for $\hat{z}_t$ reduces to the Kalman filter, and the conditional density with respect to Y_t is Gaussian. It follows that

$$\tilde{r}(t,\hat{z}_t,\Gamma_t) = \int_{-\infty}^{\infty} r(t,\theta)\frac{1}{\sqrt{2\pi\Gamma_t}} \ \mathrm{EXP}\left[-\frac{(\theta-\hat{z}_t)^2}{2\Gamma_t}\right]d\theta \ , \qquad (4\text{-}28)$$

where

$$d\hat{z}_t = \alpha\hat{z}_t dt + \Gamma_t f d\nu_t,$$

$$d\Gamma_t = (-2\alpha\Gamma_t + \beta^2 - f^2\Gamma_t^{\ 2})dt,$$

$$d\nu_t = dy_t - f\hat{z}_t dt.$$

The optimal control is given by

$$u_t^{\ o} = -\ V(t,\hat{z}_t,\Gamma_t)x_t, \qquad (4\text{-}29)$$

where $V(t,\hat{z}_t,\Gamma_t)$ is a solution to the Cauchy equation of the following form:

$$L(V) - V^2\ \tilde{r}^2 = 0, \quad V(T,\xi,\eta) = M. \qquad (4\text{-}30)$$

Here,

$$L(V) = \frac{\partial V}{\partial t} - \alpha\xi\frac{\partial V}{\partial \xi} + (-2\alpha\eta + \beta^2 - f^2\eta^2)\frac{\partial V}{\partial \eta} + \frac{1}{2}\ \eta^2 f^2\ \frac{\partial^2 V}{\partial \xi^2} \ ,$$

$$t \in [0,T], \quad \xi, \ \eta \in R.$$

Figure 4.6 shows the numerical solution to the aircraft landing problem. Here, $\alpha = -1.26$ ft/sec^2, $\beta = 1.12$ ft/sec^2, $f = 1.0$, $g = 0.0$, $r(t,z_t) = 6.37 \tan^{-1}(8.5\ z_t)$, $\Delta_\xi = 0.5$, $\Delta_\eta = 0.1$, $\Delta_t = 1.0$, $t \in [0,100]$sec, $z_0 = 1.0$, $x_0 = -15.0$ ft/sec, and $M = 0.0005$. Each Wiener process of w_t is generated from 10^2 pseudo-random Gaussian variables. The optimal-control signal and the optimum trajectory are plotted in Figures 4.6 and 4.7, respectively. For safe comfortable landing the altitude h(t) of the aircraft is often described by an exponential-linear flare path. The results shown in Figure 4.7

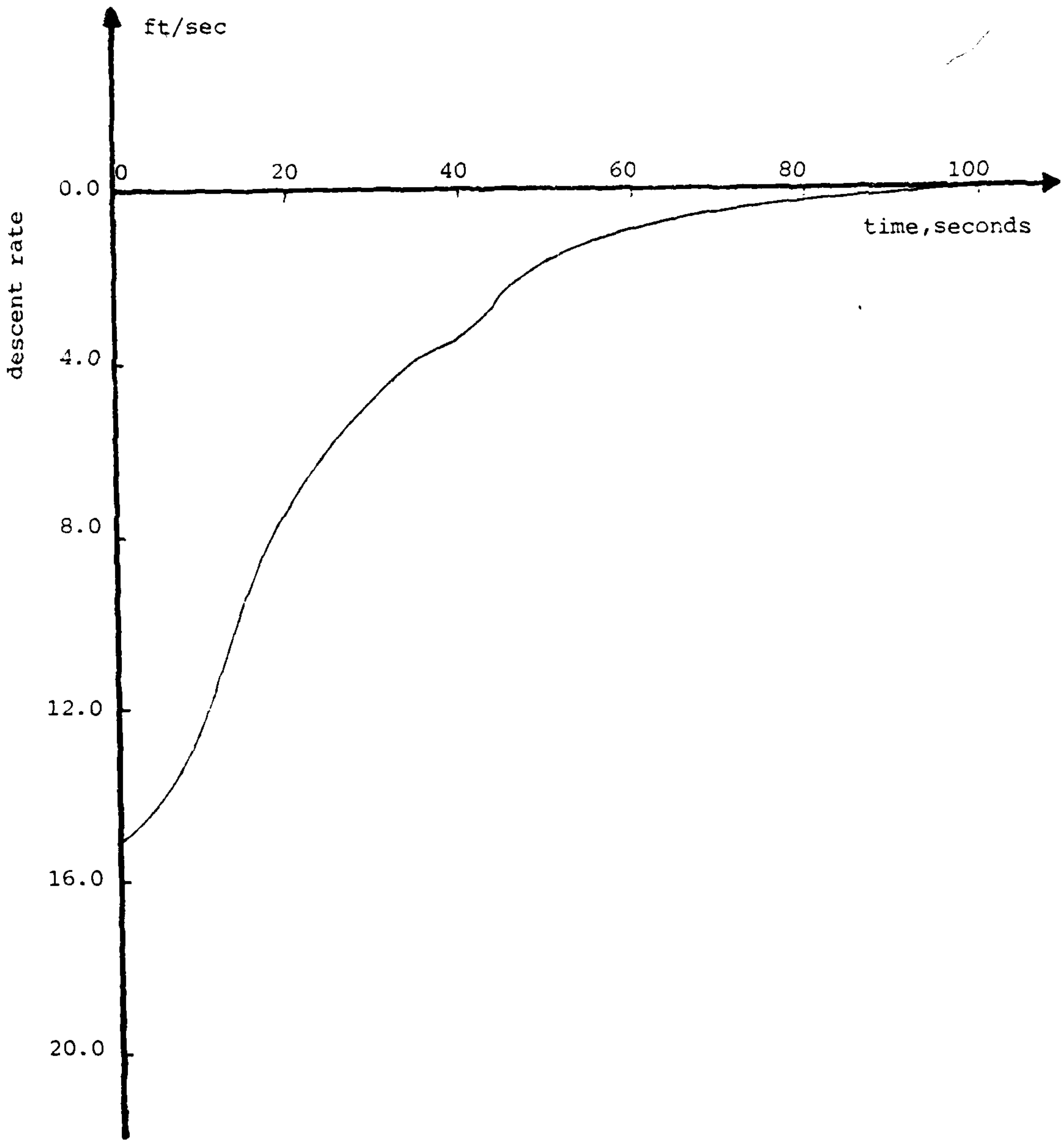

Figure 4.5 The optimum descent rate of the aircraft landing process

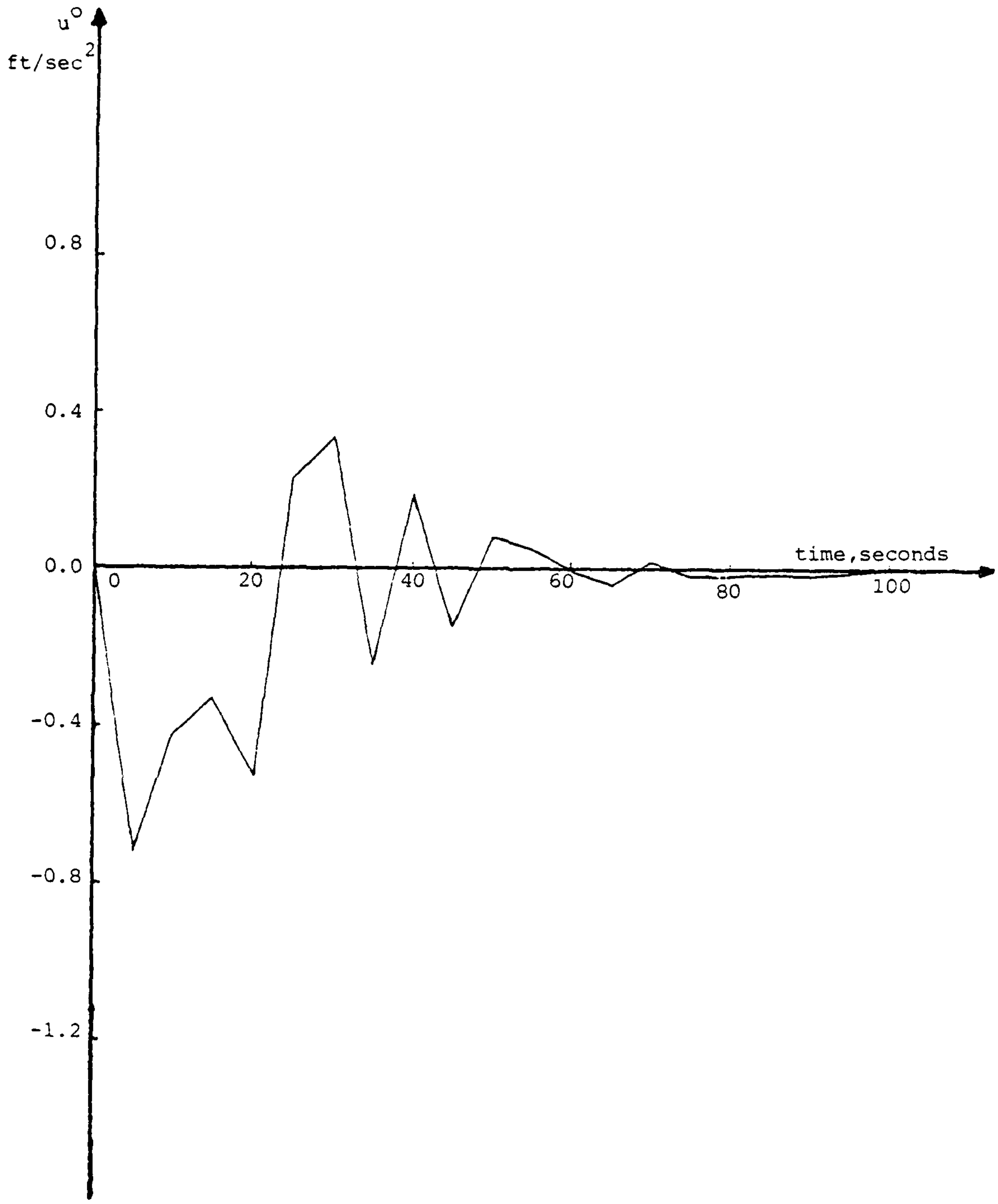

Figure 4.6 The optimal control law of the aircraft landing process

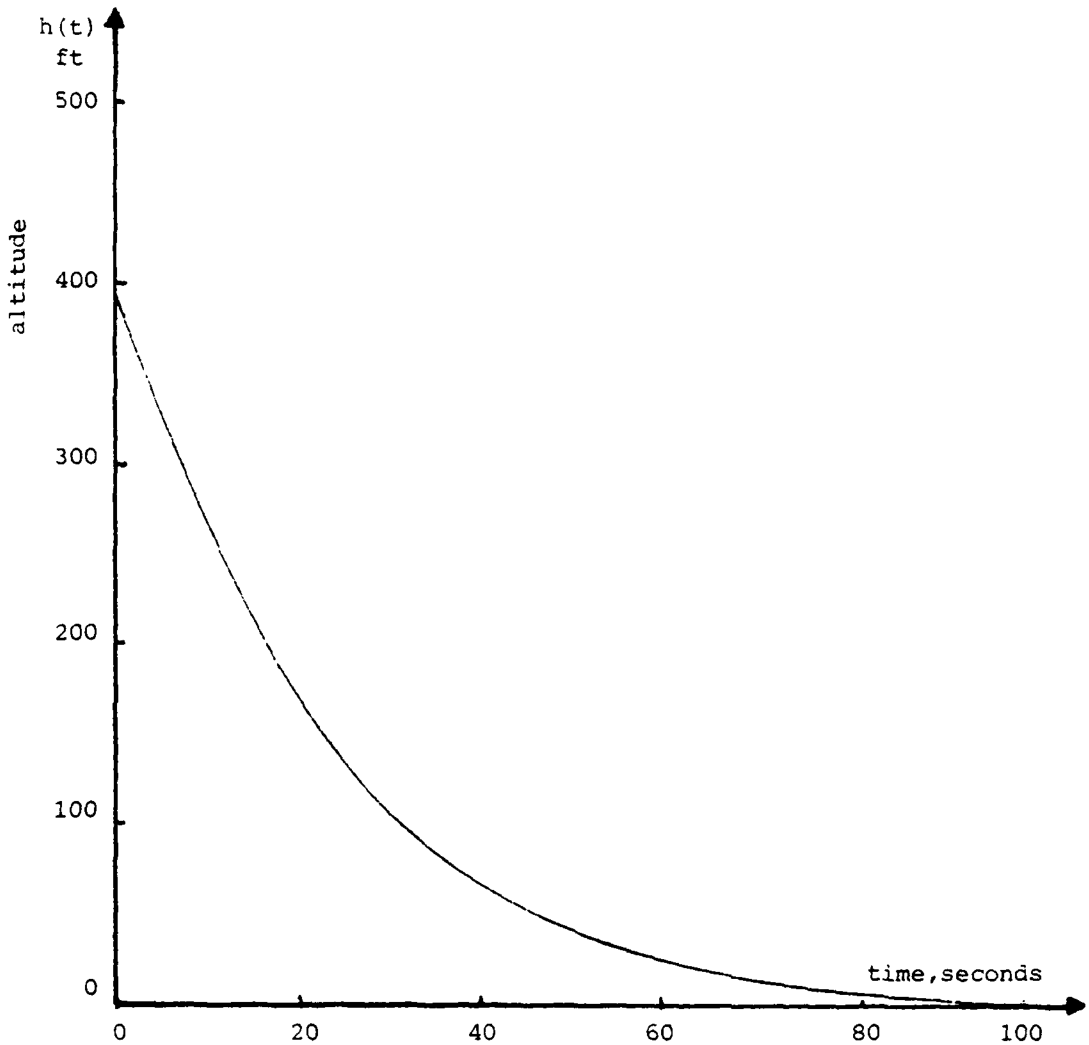

Figure 4.7 The optimum trajectory of the aircarft landing process

is almost the same as the expoential-linear flare path so popular in landing systems.

4.4 The Control Problem of Longitudinal Motion of an Aircraft in Wind Gust

The control of dynamical systems which contain uncertain elements and are subject to uncertain inputs may be treated by stochastic control theory. Leitmann [32,33] presented studies of uncertain dynamical systems which have an unstable mode, and he treated the estimated state-feedback control to assure ultimate boundedness of given states for a Lyapunov function. The purpose of this section is to develop systematic techniques for the design of an optimal-control law which will minimize the effect of the wind disturbances and noises to an aircraft in a gusty wind. The control of dynamical systems which include unknown parameters and are dependent on uncertain quantities are estimated by an appropriate filtering algorithm. Even with a stochastic model of the wind gusts, there still remains the nontrivial work of computing the statistical response of an aircraft flying through such a wind-gust field. In virtually all analyses of aircraft wind-gust response, which this author has found, the incremental aerodynamic force at each point on the aircraft are assumed to depend lineraly upon the wind-gust velocity. While such assumption may be somewhat questionable in general, its use in the Dryden model for turbulence yields a reasonably accurate description of the wind gust effects.

Consider the control problem of longitudinal motion of an aircraft in wind gust [34]. Let

x_1 = angle of attack

Θ = orientation of aircraft

$x_2 = \dot{\Theta}$

u_1 = active elevator - control angle

u_2 = active aileron - control angle

w_t = Wiener process

For such system, a stochastic dynamic equation of the motion of the aircraft is given by

$$dx_t = A(t,z_t)x_t dt + B(t,z_t)u_t dt + G(t)dw_t, \qquad (4\text{-}31)$$

$$x(0) = x_0, \quad x_t \varepsilon R^2,$$

where $A(t,z_t)$ and $B(t,z_t)$ are uncertain parameters, and w_t models the fact that the angle of attack x_1 and orientation of aircraft Θ are difficult to measure. Let the presence of the stochastic process z_t model wind gust, where z_t satisfies

$$dz_t = C(t)z_t dt + D(t)dw_t^1, \quad z(0) = z_0, \ z_t \ \varepsilon \ R^1, \qquad (4\text{-}32)$$

with observation

$$dy_t = F(t)z_t dt + H(t)dw_t^2, \ y(0) = 0, \ y_t \ \varepsilon \ R^1, \qquad (4\text{-}33)$$

where $C(t)$, $D(t)$, $F(t)$, $H(t)$ have appropriate dimensions and w_t, w_t^1, w_t^2 are independent Wiener processes.

An approximate description of (4-31) may be given by the following stochastic differential equation:

$$dx_t \cong \tilde{A}(t,\hat{z}_t, \gamma_t)x_t dt + \tilde{B}(t,\hat{z}_t,\gamma_t)u_t dt + G(t)dw_t, \qquad (4\text{-}34)$$

where $x_t^* = [x_1,x_2]$, $u_t^* = [u_1,u_2]$, and $\hat{z}_t$ is $E[z_t|\mathcal{Y}_t]$ and γ_t is $E[(z_t-z_t^*)^2|\mathcal{Y}_t]$. Here, $\tilde{A}(t,\hat{z}_t,\gamma_t)$, and $\tilde{B}(t,z_t,\gamma_t)$ are given by

$$\tilde{A}(t,\hat{z}_t,\gamma_t) = \int_{-\infty}^{\infty} A(t,\xi)\frac{1}{\sqrt{2\pi\gamma_t}} \text{EXP}\ (-\frac{1}{2}\ (\xi-\hat{z}_t)^2\gamma_t^{-1})d\xi\ ,$$

$$\tilde{B}(t,\hat{z}_t,\gamma_t) = \int_{-\infty}^{\infty} B(t,\xi)\frac{1}{\sqrt{2\pi\gamma_t}} \text{EXP}\ (-\frac{1}{2}\ (\xi-\hat{z}_t)^2\gamma_t^{-1})d\xi\ . \qquad (4\text{-}35)$$

The conditional expectation $\hat{z}_t$ and the conditional covariance γ_t are given by the following equations:

$$d\hat{z}_t = C(t)\hat{z}_t dt + \gamma_t F(t)(H(t)^2)^{-1}(dy_t - F(t)\hat{z}_t dt),$$

$$d\gamma_t = (D(t)^2 - \gamma(t)^2\ F(t)^2\ (H(t)^2)^{-1} + 2C(t)\gamma_t)dt, \qquad (4\text{-}36)$$

$$\hat{z}_0 = E(z_0)\ ,\ \gamma_0 = \text{Cov}(z_0).$$

If the state $x_t^* = [x_1,x_2]$ is observable, the minimization of the following cost function $J(u)$ can be made:

$$J(u) = E[\int_0^T (x_t^* Q x_t + u_t^* R u_t]dt\ . \qquad (4\text{-}37)$$

Then the optimal control u_t^o satisfies

$$u_t^o = -R^{-1}\tilde{B}^*(t,z_t,\gamma_t)\ (\Lambda_1(t,z_t,\gamma_t)x_t + \frac{1}{2}\ \Lambda_2(t,z_t,\gamma_t)), \qquad (4\text{-}38)$$

where $\Lambda_1(t,z_t,\gamma_t)$ and $\Lambda_2(t,z_t,\gamma_t)$ are solutions to the following nonlinear partial differential equations:

$$\dot{\Lambda}_1 = -\tilde{A}^*\Lambda_1 - \Lambda_1\tilde{A} + Q - \Lambda_1\tilde{B}R^{-1}\tilde{B}^*\Lambda_1 - (F\xi)\ \frac{\partial}{\partial\xi}\ \Lambda_1$$
$$- \frac{1}{2}\ (\eta^2 F^2 \frac{\partial^2}{\partial\xi\partial\xi}\Lambda_1) - (D^2 - \eta^2 F^2 (H^2)^{-1} + 2C\eta)\frac{\partial}{\partial\eta}\ \Lambda_1,$$

$$\dot{\Lambda}_2 = -\Lambda_2\tilde{A} - \Lambda_2\tilde{B}R^{-1}\tilde{B}^*\Lambda_1 - (F\xi)\frac{\partial}{\partial\xi}\ \Lambda_1 - \frac{1}{2}\ (\eta^2 F^2\ \frac{\partial^2}{\partial\xi\partial\xi}\ \Lambda_2)$$
$$- (D^2 - \eta^2 F^2 (H^2)^{-1})\frac{\partial}{\partial\eta}\Lambda_2. \qquad (4\text{-}39)$$

Here, the arguments (t,ξ,η) are omitted for brevity, and

$\Lambda_1(T,\xi,\eta) = 0, \quad \Lambda_2(T,\xi,\eta) = 0, \ \xi \in R^1, \ \eta \in R^1.$

For the case when the state $x_t^* = [x_1, x_2]$ is unobservable, the control problem has been solved for a linear system with quadratic criterion having random coefficients [29]. Consider the following observation equations:

$$\begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} \begin{bmatrix} dw_t^2 \\ dw_t^3 \end{bmatrix}, \qquad (4\text{-}40)$$

where w_t^2 and w_t^3, $t \in [0,T]$, are Wiener processes and are independent of w_t and w_t^1, respectively.

Let 1) to 3) in Chapter 2 be satisfied and also assume that $P[\begin{bmatrix} x_1(o) \\ x_2(0) \end{bmatrix} \leq \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \Big| \begin{bmatrix} y_1(o) \\ y_2(o) \end{bmatrix}]$ is p-a.s. Gaussian. Then m_t and Γ_t, $t \in [0,T]$, are unique, continuous, $\mathcal{Y}_t$-measurable solutions to

$$dm_t \cong \tilde{A}(t,\hat{z}_t,\gamma_t)\, m_t dt + \tilde{B}(t,\hat{z}_t,\gamma_t) u_t dt + \Gamma_t a^*(ee^*)^{-1} d\,\nu_t,$$

$$d\Gamma_t = (\tilde{A}\Gamma_t + \Gamma_t \tilde{A}^* + GG^* - \Gamma_t\, a^*(ee^*)^{-1} a\Gamma_t\,)dt, \qquad (4\text{-}41)$$

$$d\nu_t = \begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix} - \begin{bmatrix} a_1 m_1 dt \\ a_2 m_2 dt \end{bmatrix},$$

$$m_0 = E[x_0 \Big| \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}],$$

$$\Gamma_0 = E[(x_0 - m_0)(x_0 - m_0)^* \Big| \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}],$$

where ν_t is a Wiener process, and $a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$, $e = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$.

Here, the equation (4-41) stems from the equations (4-34) and (4-40).

There is a parameter of particular significance to aircraft and missiles, namely the angle of attack defined by the following relation:

$$\alpha = \tan^{-1}\left(\frac{w}{u}\right),$$

where w and u are velocities of X and Z directions, respectively. This is the angle between the relative wind and the longitudinal axis resolved into the XZ-plane. A frequently accepted alteration facilitating the mathematics is the approximation of constant forward speed. Certainly aircraft are capable of flight over a very wide speed range but the dynamic motions that are most desirable for analysis occur in a relatively short period of time - on the order of a few seconds - during which no appreciable speed change takes place. In face axial accelerations of the order of which aircraft are usually subjected do not significantly affect their dynamic motion. Under this assumption, the angle of attack is

$$\alpha = \tan^{-1}\left(\frac{w}{\text{constant}}\right).$$

Aerodynamic forces and moments are strictly speaking functions of certain state variables. Consider for example variable angle of attack $\alpha(t)$, $-\pi < \alpha(t) \leq \pi$, on wing with velocity w(t) and u(t). Using aerodynamic derivatives or stability derivates [78] with respect to the longitudinal-motion variables, expressions of

$A(t,z_t)$ in equation (4-31) for the aerodynamic forces and moments may be approximated by $A(t,\tan^{-1}z_t)$.

Computations are carried out for the following:

$$A = \begin{bmatrix} -2.03 - 1.015\tan^{-1}z_t & 1.00 \\ -8.57 - 4.285\tan^{-1}z_t & -2.75 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.021 & -0.156 \\ -1.82 & -0.550 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.012 \\ 0.051 \end{bmatrix}, \quad a = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix}, \quad e = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix},$$

$$C(t) = -1.256673, \qquad D(t) = 1.1209982,$$

$$F(t) = 1.0, \qquad H(t) = 1.0,$$

$$z(0) = 0.1 \qquad t \in [0,10]\,\text{sec},$$

$$Q = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \qquad R = \begin{bmatrix} R_1 & 0.0 \\ 0.0 & R_2 \end{bmatrix}.$$

The cost function is chosen for approximating the aircraft's weight and for control. For the case of longitudinal motion of an aircraft in wind gusts, a reasonable choice for the cost function is choosing the weight factors as

$$\frac{1}{R_1} = \pi \text{ rad sec}^{-1}, \quad \frac{1}{R_2} = \pi \text{ rad sec}^{-1}.$$

Subject to the constraint of aircraft dynamics, it is assumed that

$$x_1(0) = 0.085 \text{ rad}, \quad x_2(0) = 0.01 \text{ rad}.$$

Figure 4.8 shows the solutions of nonlinear partial differential

equations for the value function $V = x_t^* \Lambda_1 x_t + \Lambda_2$. Each trajectory is given for the three solutions and each solution is subject to uncertain random parameter z_t which is approximated by the conditional estimate $E[z|Y_t]$ and covariance $E[(z_t - \hat{z}_t)^2|Y_t]$.

The attack angle relative to unperturbed air is determined corresponding to optimal control laws and is plotted in Figure 4.9. For the attack angle x_1, Figure 4.9(a) presents the observable case, and (b) presents the unobservable case, respectively. The state of orientation rate of aircraft relative to inertial line is plotted in Figure 4.10. The state trajectory is given for two cases. Figure 4.10(a) shows the orientation range rate for observable and Figure 4.10(b) unobservable cases.

The results of the optimal-control-law calculations are presented in Figures 4.11 and 4.12. As will be seen by comparing Figure 4.11(a) and (b), the active elevator angle of the aircraft for the observable x_t and unobservable x_t, respectively, are presented. Finally a comparison is depicted in Figure 4.14 to the active aileron control angle. Figure 4.12(a) and (b) also show the aileron control angle for the observable x_t and unobservable x_t, respectively. Those comparisons are found to be even more smooth in the conditional-Gaussian filtering for the unobservable state x_t.

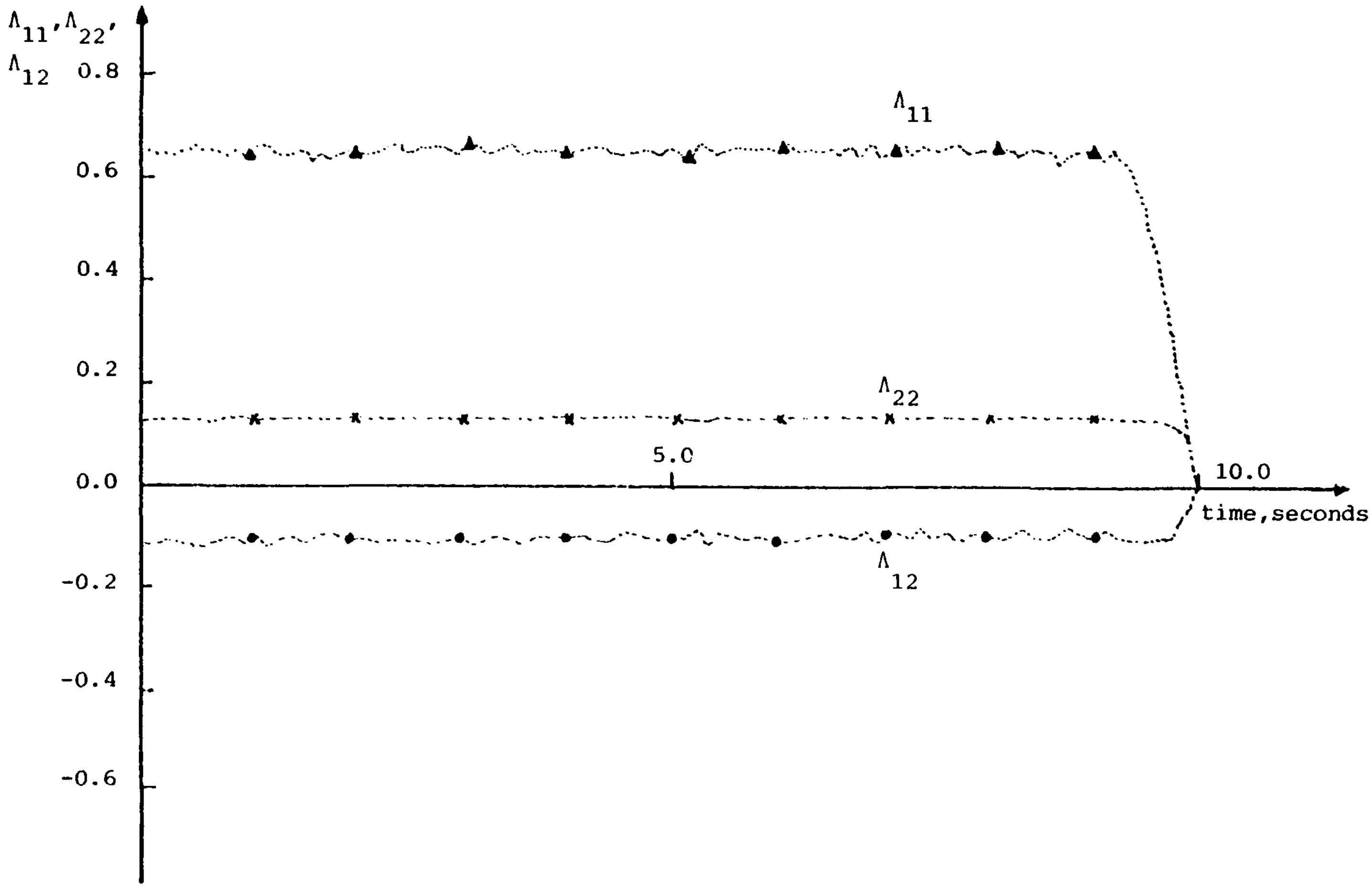

Figure 4.8 Solutions of Riccati-like-equation for the control of longitudinal motion of an aircraft

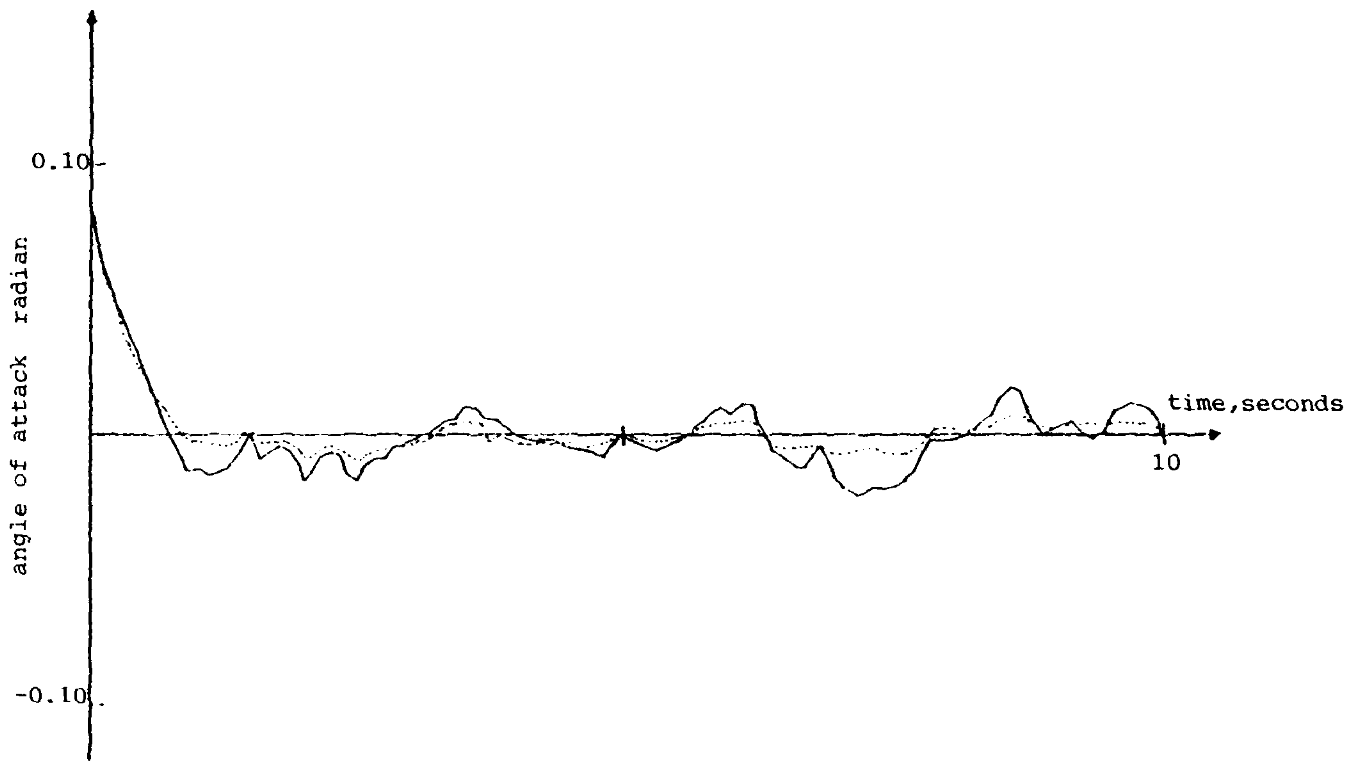

Figure 4.9 The optimum trajectory of (a) observable (——) *and (b) unobservable* (····) angle of attack

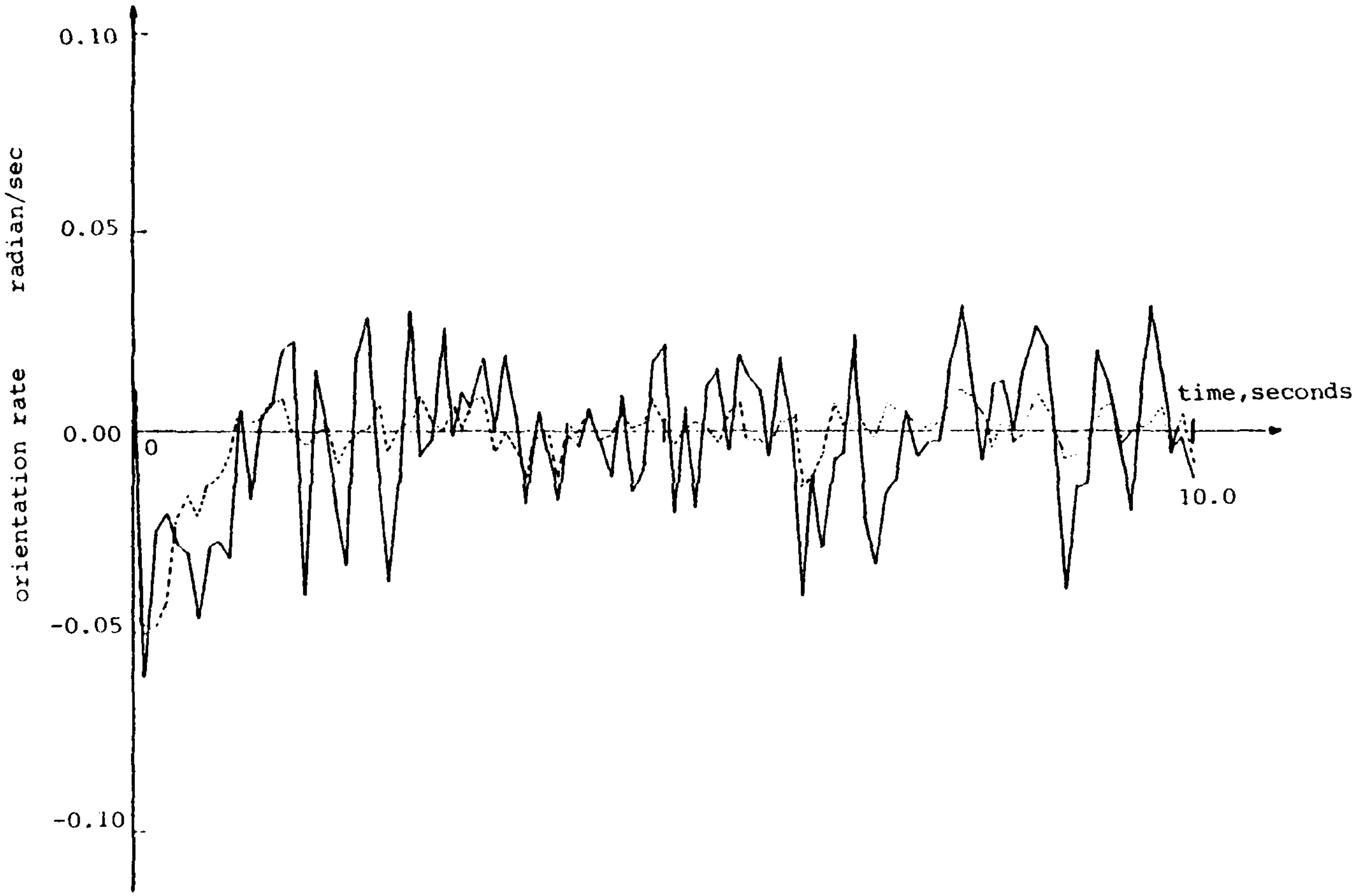

Figure 4.10 The optimum trajectory of (a) observable (——) and (b) unobservable (····) orientation range rate

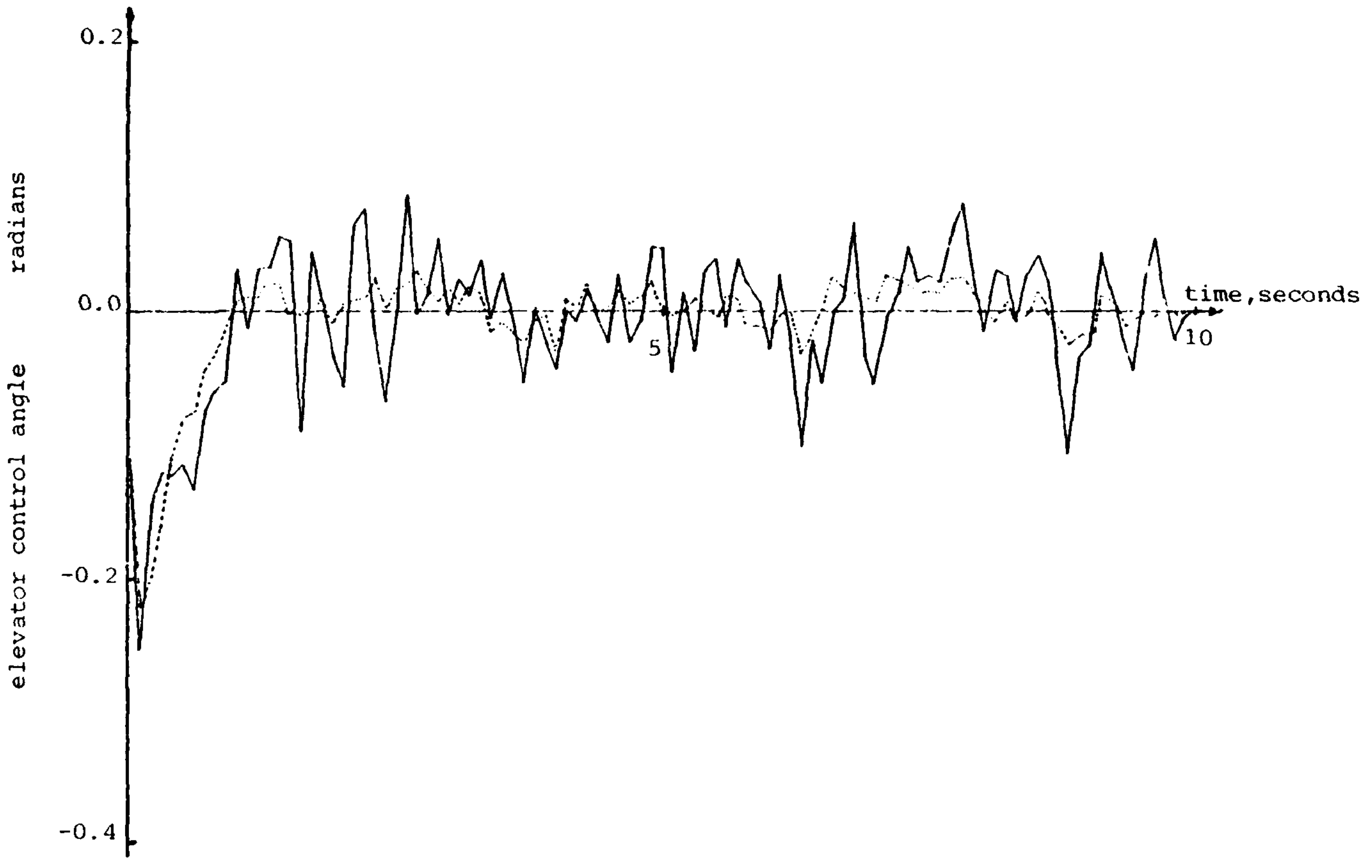

Figure 4.11 Optimal control of elevation control angle for (a) observable states (___) and (b) unobservable states(....)

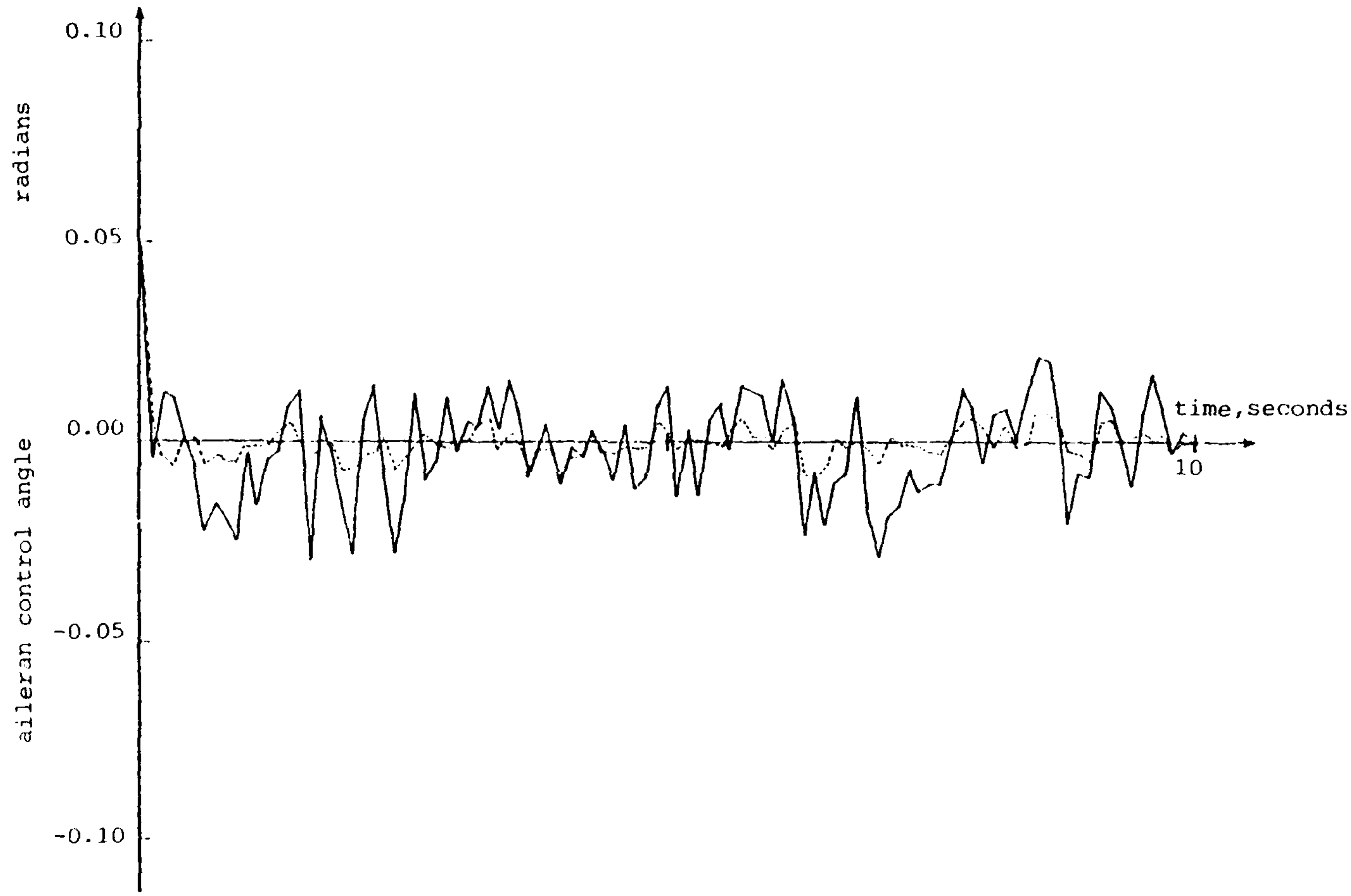

Figure 4.12 Optimal control of aileron control angle for (a) observable states (——) and (b) unobservable states (····)

5. ON NONLINEAR FILTERING AND TRACKING OF TARGET

Among anti-submarine target-motion analysis algorithms for mobile platforms, one convenient mathematical tracking model consists of describing the target dynamics by a set of state-variable equations that are driven by a Wiener process. Dynamic motion of the moving target is generally nonlinear and unobservable; in passive bearings-only-tracking problems the observations usually appear in the argument of an arc-tangent function. The extended Kalman filter is a popular method for treating such nonlinear estimation problems [47,48,51,52,55]. However, if nonlinearities are sufficiently important, the estimation error can be significantly reduced through use of a higher-order estimation technique.

It is the purpose of this chapter to compare performance between an extended Kalman filter and a truncated second-order nonlinear filter as applied to bearings-only-target tracking using simulation studies. An approach to tracking a maneuvering target is also considered.

5.1 Modeling in Two-dimensional Space

A two-dimentionsl version of the bearings-only-tracking problem is considered. A target O is moving in relation to an observer in a viscous fluid as shown in Figure 5.1. The control U_{T_1} and U_{T_2} are thrusts in the X and Y direction, respectively. Assume that the magnitude of drag force is $\alpha \cdot$ [velocity of O]^2 [53,54] where α is a drag coefficient; the direction is opposite

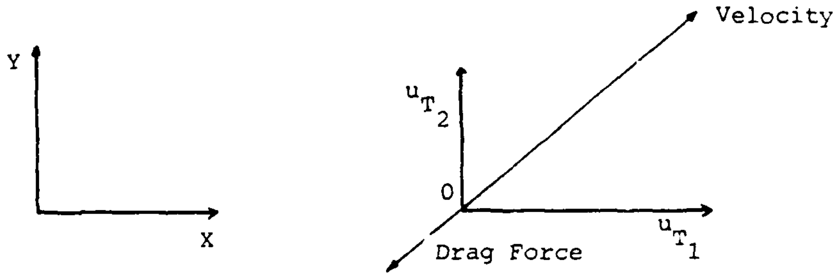

Figure 5.1 Geometrical definition of vector tracking

to the instantaneous velocity vector. Under the assumptions of planar motion and constant mass, define $x_1 \triangleq X$, $x_2 \triangleq \dot{X}$, $x_3 \triangleq Y$, $x_4 = \dot{Y}$, the state equations of the target motion may be approximated by

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t),\\
\dot{x}_2(t) &= -\alpha x_2(t)\sqrt{x_2^2(t)+x_4^2(t)} + u_{T_1},\\
\dot{x}_3(t) &= x_4(t),\\
\dot{x}_4(t) &= -\alpha\, x_4(t)\sqrt{x_2^2(t)+x_4^2(t)} + u_{T_2},
\end{aligned} \tag{5-1}$$

with an observed process y(t) of the form

$$y(t) = \tan^{-1}\frac{x_3(t)}{x_1(t)}\,. \tag{5-2}$$

Consider from (5-1) and (5-2), the following stochastic differential equation:

$$dx(t) = f(x(t),t)dt + B(t)u(t)dt + G(t)dw_t, \tag{5-3}$$

and an observed stochastic process, y(t) such that

$$y(t) = h(x(t),t) + R(t)v_t, \tag{5-4}$$

where the dimensions of (5-3) are equivalent to that of (5-1), and w_t^1, w_t^2, v_t, are mutually independent Wiener processes.

Here,

$$f(x(t),t) = \begin{bmatrix} x_2(t) \\ -\alpha x_2(t)\sqrt{x_2^2(t)+x_4^2(t)} \\ x_3(t) \\ -\alpha x_4(t)\sqrt{x_2^2(t)+x_4^2(t)} \end{bmatrix},$$

$$[B(t)\ u(t)]^* = [\ 0 \quad u_{T_1} \quad 0 \quad u_{T_2}\ ],$$

$$[G(t)\ dw_t] = \begin{bmatrix} 0 & 0 \\ e_1(t) & 0 \\ 0 & 0 \\ 0 & e_2(t) \end{bmatrix} \begin{bmatrix} dw_t^1 \\ dw_t^2 \end{bmatrix},$$

and

$$h(x(t),t) = \tan^{-1}\frac{x_3(t)}{x_1(t)}.$$

Given the stochastic equations of target motion and measurement information in (5-3) and (5-4), consider algorithms for calculating the minimum-variance estimate of $x(t)$ as a function of time and the accumulated measurement data. In this case the extended-Kalman filter equation for (5-3) is given by

$$d\hat{x}(t) = f(\hat{x}(t),t)dt + B(t)u(t)dt + P(t)h_x^*(\hat{x}(t),t)(RR^*)^{-1}(y(t)dt - h(\hat{x}(t),t)dt), \tag{5-5}$$

$$\dot{P}(t) = f_x(\hat{x}(t),t)P(t) + P(t)f_x^*(\hat{x}(t),t) + G(t)G(t)^*$$
$$-P(t)h_x^*(\hat{x}(t),t)(RR^*)^{-1}h_x(\hat{x}(t),t)P(t),$$
$$\hat{x}(0) = E[x(0)] = x_0, \quad P(0) = \mathrm{Cov}[x_0 x_0^*], \tag{5-6}$$

where, for any integrable random process $x(t)$, $f_x(\cdot) = \frac{\partial f(\cdot)}{\partial x^*}$ and denote $E[x(t)|y(s),\ 0 \leq s \leq t]$ by $\hat{x}(t)$ or $E_t[x(t)]$ and $P(t)$ is the error covariance matrix. The extended Kalman filter is a useful method for considering nonlinear estimation problems. If nonlinearities are important, the unobservable state-estimation error can be reduced through use of higher-order estimation methods. Simulation experiments with the truncated, second-order, nonlinear filter [55, 79,80] clearly show that it may improve the estimates compared to the extended Kalman filter.

In general, the conditional-mean estimate $E_t[x(t)]$ satisfies the stochastic equation

$$dE_t[x(t)] = E_t[f(x(t),t)]dt + B(t)u(t)dt$$
$$+ \{E_t[x(t)h^*(x(t),t)]$$
$$- E_t[x(t)]E_t[h^*(x(t),t)]\}(RR^*)^{-1}d\nu_t, \tag{5-7}$$

where the innovations process corresponding to (5-7) is

$$\nu_t = y(t) - E_t[h(x(t),t)]. \tag{5-8}$$

The high-order estimation equation for (5-7) can be approximately obtained as follows. The truncated second-order nonlinear filter was developed by Bass, et al., [79], and independently Jazwinski [55]. This filter carries to second order, whereas third and higher order central moments are neglected. Recently Henriksen [80]

rederived the truncated second-order nonlinear filter. The following truncated second-order filter estimate $\hat{x}(t)$ in (5-5) and (5-6) satisfies

$$
\begin{aligned}
d\hat{x}(t) = {} & f(\hat{x}(t),t)dt + \frac{1}{2} f_{xx}(\hat{x}(t),t)[CM(P(t))]dt + B(t)u(t)dt \\
& + P(t)h_x^*(\hat{x}(t),t)(RR^*)^{-1}\{y(t)dt - [h(\hat{x}(t),t)dt \\
& + \frac{1}{2} h_{xx}(\hat{x}(t),t)[CM(P(t))]]dt\}
\end{aligned}
\tag{5-9}
$$

$$
\begin{aligned}
\dot{p}(t) = {} & f_x(\hat{x}(t),t)P(t) + P(t)f_x^*(x(t),t) + G(t)G(t)^* \\
& - P(t)h_x^*(\hat{x}(t),t)(RR^*)^{-1}h_x(\hat{x}(t),t)P(t),
\end{aligned}
\tag{5-10}
$$

where $f_x(\cdot) = \partial f(\cdot)/\ \partial x^*$, whereas $f_{xx}(.,.)$ is the 2×2^2 matrix given by

$$
f_{xx} = \frac{\partial^2 f}{\partial x \partial x^*} = \frac{\partial}{\partial x^*}(f_x) = [\frac{\partial f_x}{\partial x_1} \quad \frac{\partial f_x}{\partial x_2} \quad \frac{\partial f_x}{\partial x_3} \quad \frac{\partial f_x}{\partial x_4}] ,
$$

And CM(A) denotes $[A_{c_1}^* \quad A_{c_2}^* \quad . \ . \ . \quad A_{c_n}^*]$ for an arbitrary matrix A where A_{c_i} denotes the ith column.

5.2 On the Maneuvering Target Problem

Consider the target free to maneuver with a finite continuous range of acceleration from which a pilot can choose. Earlier work on this area included Jazwinski's work on the limited-memory filtering which will respond faster than the normal filter [55]. Thorp described a technique of switching between two Kalman filters, one of which had a fixed gain in order to determine when a maneuver occurs [56]. The deterministic maneuver command u_k, $k = 1,2,\ldots,n$ modeled as a semi-Markov process was introduced by Moose [48,49].

The work of Sworder, et al., [57,58] is aimed primarily at the development of an adaptive control system for linear systems in which parameters may jump abruptly. Friedland introduced properties of the separate-bias estimation technique including the interpretation of the result as the estimation of a constant embedded in white noise [59]. This method of separating the estimation of the bias variables that are unknown constants from the linear dynamic variable was presented. Another useful approach was presented by Mehra, et al., [60], Wilsky, et al., [61], and Chan, et al., [62] using evaluation of the properties of the *residual sequence (innovations process).*

Suppose this target is maneuvering; then one can examine the possibility of the target for velocity change and course change during the certain small time interval Δt. Since the maneuvering is unknown, it makes sense, from a knowledge of the performance capabilities and operating patterns of target, to analyze the measurement information for observations given assumptions and current estimate of all relevant quantities. If the target is maneuvering at an average speed in a certain direction, the acceleration forces u(t) must be equal to drag forces to each direction, or

$$u_{T_1}(t) = \alpha\, E_t[x_2(t)\sqrt{x_2^{\,2}(t)+x_t^{\,2}(t)}] \cong \alpha\, \hat{x}_2(t)\sqrt{\hat{x}_2^{\,2}(t)+\hat{x}_4^{\,2}(t)}\ ,$$

$$u_{T_2}(t) = \alpha\, E_t[x_4(t)\sqrt{x_2^{\,2}(t)+x_4^{\,2}(t)}] \cong \alpha\, \hat{x}_4(t)\sqrt{\hat{x}_2^{\,2}(t)+\dot{x}_4^{\,2}(t)}\ .$$

The filter performs well if there are no modeling errors; then the computed error covariance P(t) becomes small, and the filter

relies on old measurements for its estimates and is oblivious to new measurements. If an abrupt change occurs, the filter will respond quite sluggishly, yielding poor performance. Consequently, the present filter has to be sensitive to new data so that maneuvering of target will be reflected in the filter behavior. Now extend the model to the case when there exists the maneuver of target, then

$$u(t) = \begin{bmatrix} u_{T_1} + \xi_1 u_{a_1} \\ u_{T_2} + \xi_2 u_{a_2} \end{bmatrix},$$

where u_{a_1} and u_{a_2} are accelerations in X and Y directions, respectively. The new target motion equation for maneuvering is given by

$$\begin{bmatrix} dx_1(t) \\ dx_2(t) \\ dx_3(t) \\ dx_4(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\alpha\, x_2(t)\sqrt{x_2^2(t)+x_4^2(t)} \\ x_4(t) \\ -\alpha\, x_4(t)\sqrt{x_2^2(t)+x_4^2(t)} \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} u_{T_1} \\ u_{t_2} \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ \xi_1 & 0 \\ 0 & 0 \\ 0 & \xi_2 \end{bmatrix} \begin{bmatrix} u_{a_1} \\ u_{a_2} \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ G_1(t) & 0 \\ 0 & 0 \\ 0 & G_2(t) \end{bmatrix} \begin{bmatrix} dw_t^1 \\ dw_t^2 \end{bmatrix}, \tag{5-11}$$

where ξ_1 and ξ_2 are certain random variables. The equation (5-11) is a bilinear stochastic system. Here

$$\xi_1(t) = \begin{cases} 0 & t < T \\ 1 & t \geq T \end{cases}, \qquad \xi_2(t') = \begin{cases} 0 & t' < T' \\ 1 & t' \geq T' \end{cases}$$

and T and T´ are random variables. Suppose that T,T´ are exponentially distributed with lifetimes λ and λ', and independent of x_0, w_t, v_t. Process transitions are jumps so that between jumps it remains in specific states at random times T,T´. For the linear dynamic case Davis [63] derives the optimal, infinite-dimensional equations for the computation of the conditional mean of x(t) and the conditional probability

$$\mathcal{E}_t \begin{bmatrix} [\xi_1 \mid y(s),\ 0 \le s < t] \\ [\xi_2 \mid y(s'),\ 0 \le s' < t'] \end{bmatrix} = \begin{bmatrix} \Pr[t \ge T \mid y(s), 0 \le s < t'] \\ \Pr[t' \ge T' \mid y(s'), 0 \le s \le t'] \end{bmatrix}$$

where T and T´ are random variables, respectively. In this case, jump processes have the difficulties for the problem of detection of jumps.

Consider the dynamic process (5-11) where u_{a_1} and u_{a_2} are random processes but comprise an unknown bias vector and depend upon the following stochastic equation:

$$\begin{bmatrix} du_{a_1} \\ du_{a_2} \end{bmatrix} = \begin{bmatrix} G_3(t) & 0 \\ 0 & G_4(t) \end{bmatrix} \begin{bmatrix} dw_t^3 \\ dw_t^4 \end{bmatrix} \tag{5-12}$$

where w_t^3, w_t^4 and mutually independent Wiener processes. Here, $G_3(t)$ and $G_4(t)$ are zero, and u_{a_1} and u_{a_2} are constant unknown biases. Under the consideration of (5-12), the maneuvering target dynamic equation is approximated by

$$\begin{bmatrix} dx_1(t) \\ dx_2(t) \\ dx_3(t) \\ dx_4(t) \\ du_{a_1}(t) \\ du_{a_2}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\alpha\ x_2(t)\sqrt{x_2^{\,2}(t)+x_4^{\,2}(t)}+u_{a_1}(t) \\ x_4(t) \\ -\alpha\ x_4(t)\sqrt{x_2^{\,2}(t)+x_4^{\,2}(t)}+u_{a_1}(t) \\ 0 \\ 0 \end{bmatrix} dt$$

$$+ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_{T_1} \\ u_{T_2} \end{bmatrix} dt$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ G_1(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & G_2(t) & 0 & 0 \\ 0 & 0 & G_3(t) & 0 \\ 0 & 0 & 0 & G_4(t) \end{bmatrix} \begin{bmatrix} dw_t^{\,1} \\ dw_t^{\,2} \\ dw_t^{\,3} \\ dw_t^{\,4} \end{bmatrix} . \qquad (5\text{-}13)$$

And observation is given by

$$y(t) = h(x(t),t) + Rv_t \qquad (5\text{-}14)$$

Here, also assumed $w_t^{\,1}$, $w_t^{\,2}$, $w_t^{\,3}$, $w_t^{\,4}$, v_t are mutually independent.

Let (5-13) denote

$$dz(t) = f'(z(t),t)dt + B'(t)u(t)dt + G'(t)dw_t' \quad (5\text{-}15)$$

Application of the extended-Kalman-filtering theory given observation (5-14) to the process governed by (5-15) results in the following equations:

$$d\hat{z}_t = \begin{bmatrix} d\hat{x}(t) \\ d\hat{u}_a(t) \end{bmatrix} = f'(\hat{z}(t),t)dt + B'(t)u_T dt \quad (5\text{-}16)$$

$$+ \gamma(t)h_x^*(\hat{x}(t),t)(RR^*)^{-1}(y(t)dt-h(\hat{x}(t),t)dt),$$

$$\dot{\gamma}(t) = f_z'(\hat{z}(t),t)\gamma(t) + \gamma(t)f_z'^*(\hat{z}(t),t) + G'(t)G'(t)^* \quad (5\text{-}17)$$

$$- \gamma(t)h_x^*(\hat{x}(t),t)(RR^*)^{-1}h_x(\hat{x}(t),t)\gamma(t).$$

The covariance matrix $\gamma(t)$ in (5-17) is partitioned as follows;

$$\gamma(t) = \left[\begin{array}{c|c} P_x & P_{xu_a} \\ \hline P_{xu_a}^* & P_{u_a} \end{array}\right] \quad (5\text{-}18)$$

P_x = autocovariance of estimate of state $x(t)$

P_{u_a} = autocovariance of acceleration estimate of acceleration $u_a(t)$

P_{xu_a} = crosscovariance of $x(t)$ and $u_a(t)$

In terms of the submatrix of (5-18), the variance equation (5-17) consists of the following three forms

(5-19)

$$\dot{P}_x(t) = f_x(\hat{x}(t),t)P_x(t) + P_x(t)f_x^*(\hat{x}(t),t) + b(t)P_{xu_a}^* + P_{xu_a}b^*(t)$$

$$-P_x(t)h_x(\hat{x}(t),t)(RR^*)^{-1}h_x(\hat{x}(t),t)\ P_x(t) + G(t)G(t)^*,$$

$$\dot{P}_{u_a}(t) = g(t)g^*(t) - P^*_{xu_a}(t)h^*_x(\hat{x}(t),t)(RR^*)^{-1}h_x(\hat{x}(t),t)P_{xu_a}(t), \tag{5-20}$$

$$\dot{P}_{xu_a}(t) = f_x(\hat{x}(t),t)P_{xu_a}(t) + b(t)P_{u_a}(t)$$
$$-P_x(t)h^*_x(\hat{x}(t),t)(RR^*)^{-1}h_x(\hat{x}(t),t)P_{xu_a}(t), \tag{5-21}$$

where

$$b^*(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad g(t) = \begin{bmatrix} G_3(t) & 0 \\ 0 & G_4(t) \end{bmatrix}.$$

If there is no bias, i.e., no maneuvering, the variance equation (5-19) is the same as (5-6), and $P_{xu_a}(t) = 0$ and $P_{u_a}(t) = 0$. Mendel, et al., [81] show that the estimation of the bias vector $u_a(t)$ can be interpreted as being equivalent to the estimation of a constant that is observed through white noise. However, for the dynamic model equation in (5-13) observations do not have any terms of unknown maneuvering bias vector u_a except the certain function of state x(t). Hence the suboptimal estimate can be found by solving (5-16) and (5-17). If there exists maneuvering of the target, (5-19), (5-20), and (5-21) are all coupled and hence have to be solved together.

5.3 Simulation Experiment of Moving Target

Initially the target starts at X=10,000 feet and Y=10,000 feet. Assume that the target maintains an average speed of $20\sqrt{2}$ ft/sec on a course 45-degrees and all initial values of estimates replace 20 different normal random numbers given 1 percent of the above

specified values. All the computer runs are made with 20 different runs to obtain the root-mean-square-error statistics associated with each geometry. The results present for root-mean-square velocity and position to the extended Kalman filter and the truncated second-order filter, respectively. The results of the estimation processes are presented by Figure 5.2, 5.3, 5.4. In the weighting of the observation data it is assumed that the noise level on the bearing information is 0.05 radians on the bearing observations. The dynamical noise levels are 1 percent of velocity. The simulation results presented here demonstrate that the extended Kalman filter and truncated second-order filter are stable estimation algorithms forbearing only target motion analysis. Furthermore, Figure 5.4 reveals that the truncated second-order filter is much better than the extended Kalman filter for the time duration of the first 15 seconds. After 15 seconds the extended Kalman filter performs better than the truncated second-order filter because the higher nonlinearity in (5.3) disappears. It means that the target motion has reached steady state after about 15 seconds.

The second simulation example is similar to the first, the target maintains an average speed of $20\sqrt{2}$ ft/sec. Here if it is assumed that all system parameters are given constants, the starting time for turns and general maneuvering can be detected from the innovations process. All conditions are the same as the first, and assume $x_5(0) = 0$, $x_6(0) = 0$ ft/sec^2 in equation (5-13).

According to equation (5-13) and (5-14), all cases are tested with the extended Kalman filter and the truncated second-order

filter. The results of a fast 90-degree right turn and the maneuvering target for the new average speed $30\sqrt{2}$ ft/sec after about 50 seconds are presented in Figures 5.5, 5.6, 5.7. In this plot, the exact target track, and the estimated track using the extended Kalman filter and the truncated second-order filter, respectively are shown as the estimate X, Y ranges of each direction, and the estimated range of the target. The presence of target-estimation delay to compare with the exact target track is visible. The simulation results show that with abrupt changes in bearing and velocity of a target, the results of the extended Kalman filter according to equation (5-13) and (5-14) indicate the effectiveness of the new technique which promises improved tracking for maneuvering target.

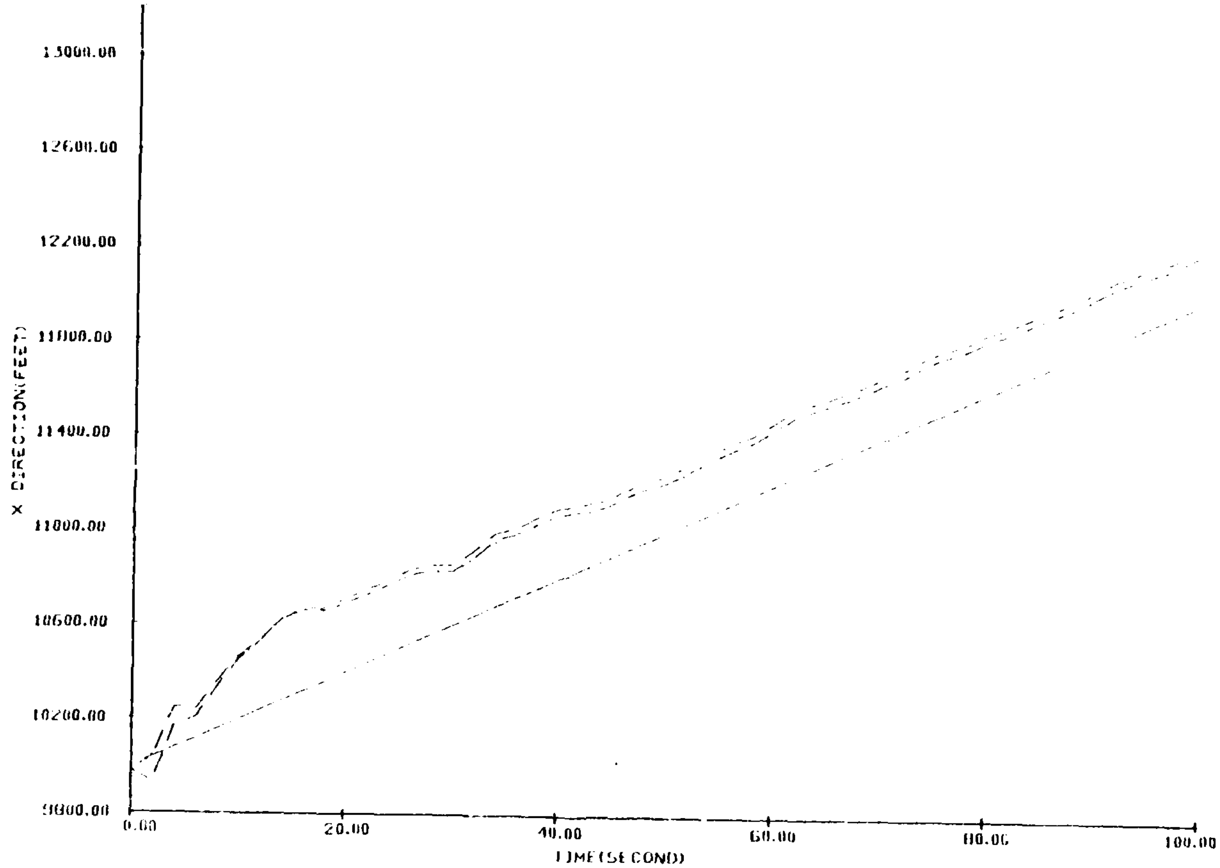

Figure 5.2 Ranges of X-direction of (a) extended Kalman filter (-·-·-), (b) truncated second-order filter (---), and (c) reference trajectory (——) for constant speed of target

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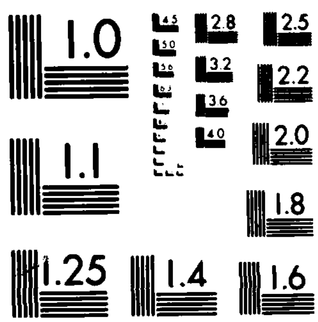

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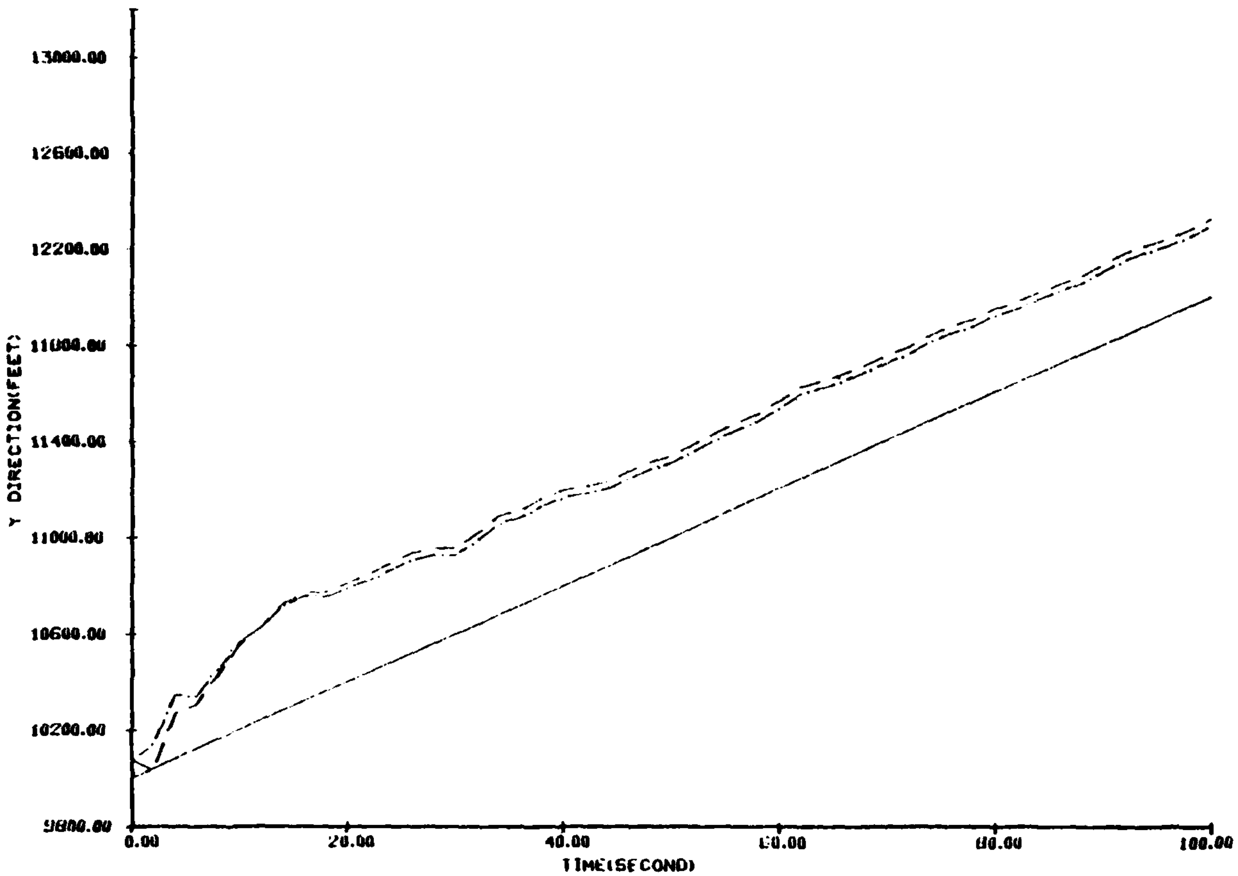

Figure 5.3 Ranges of Y-direction of (a) extended Kalman filter (-·--·-), (b) truncated second-order filter (---), and (c) reference trajectory (——) for constant speed of target

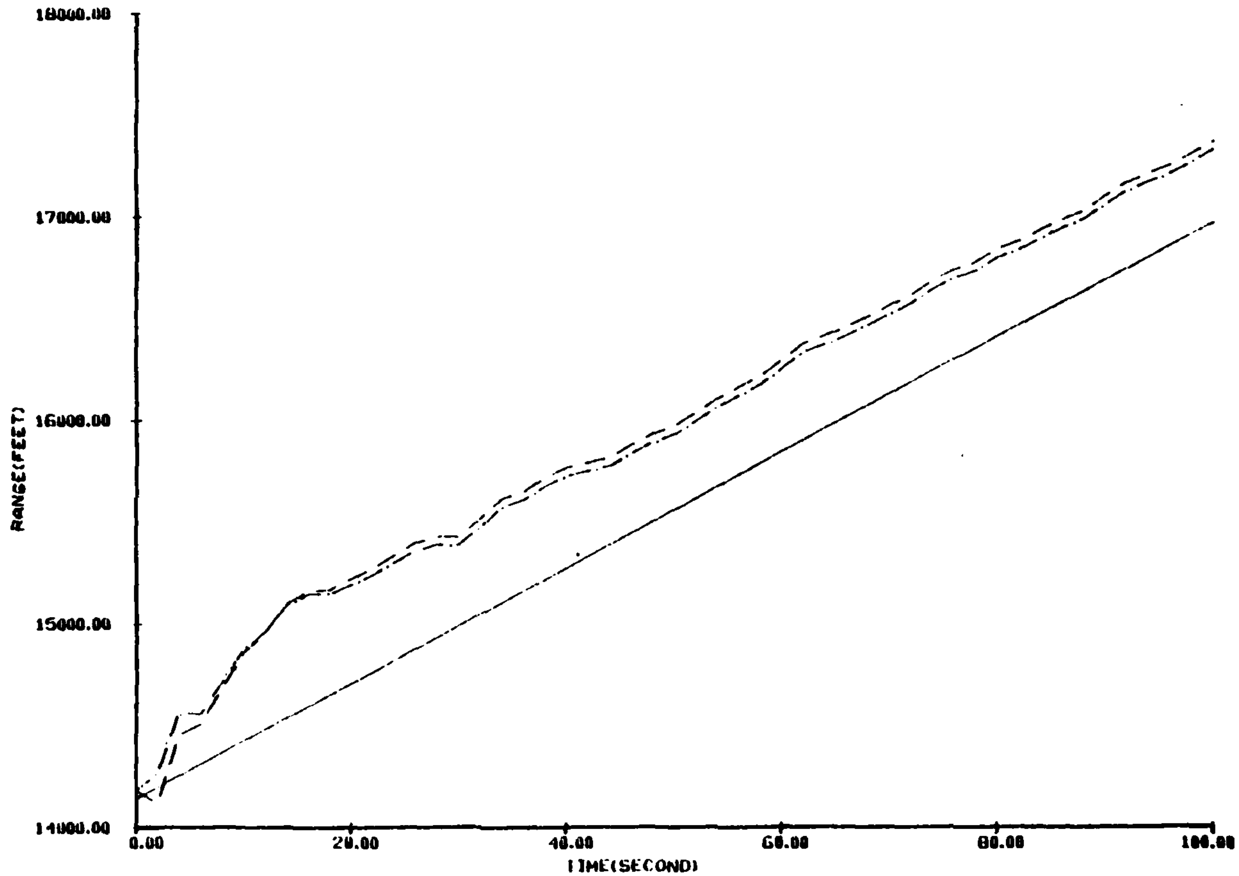

Figure 5.4 Ranges of (a) extended Kalman filter (·-·-·-), (b) truncated second-order filter, and (c) reference trajectory (——) for constant speed of target

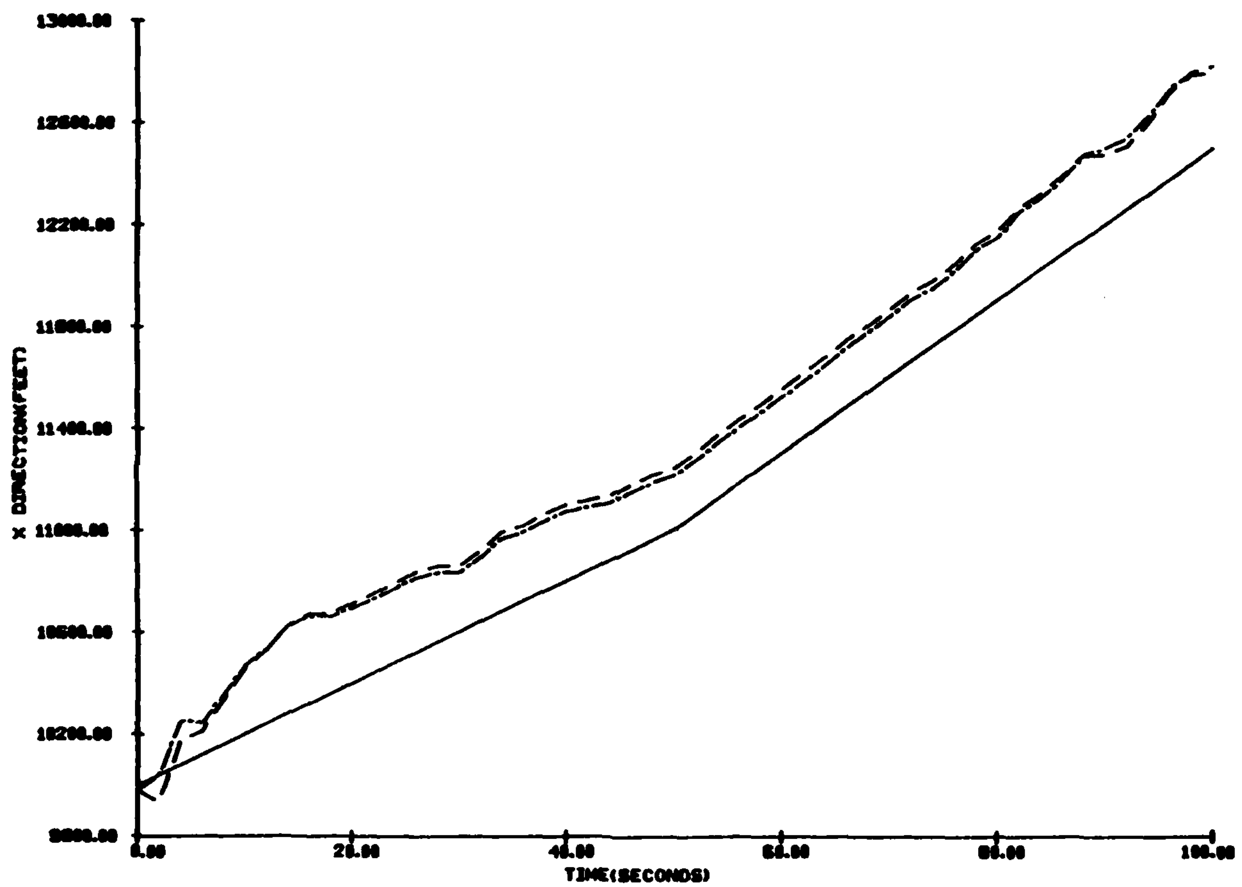

Figure 5.5 Ranges of X-direction of (a) extended Kalman filter (-·-·-), (b) truncated second-order filter (----), and (c) reference trajectory (——) for maneuvering target

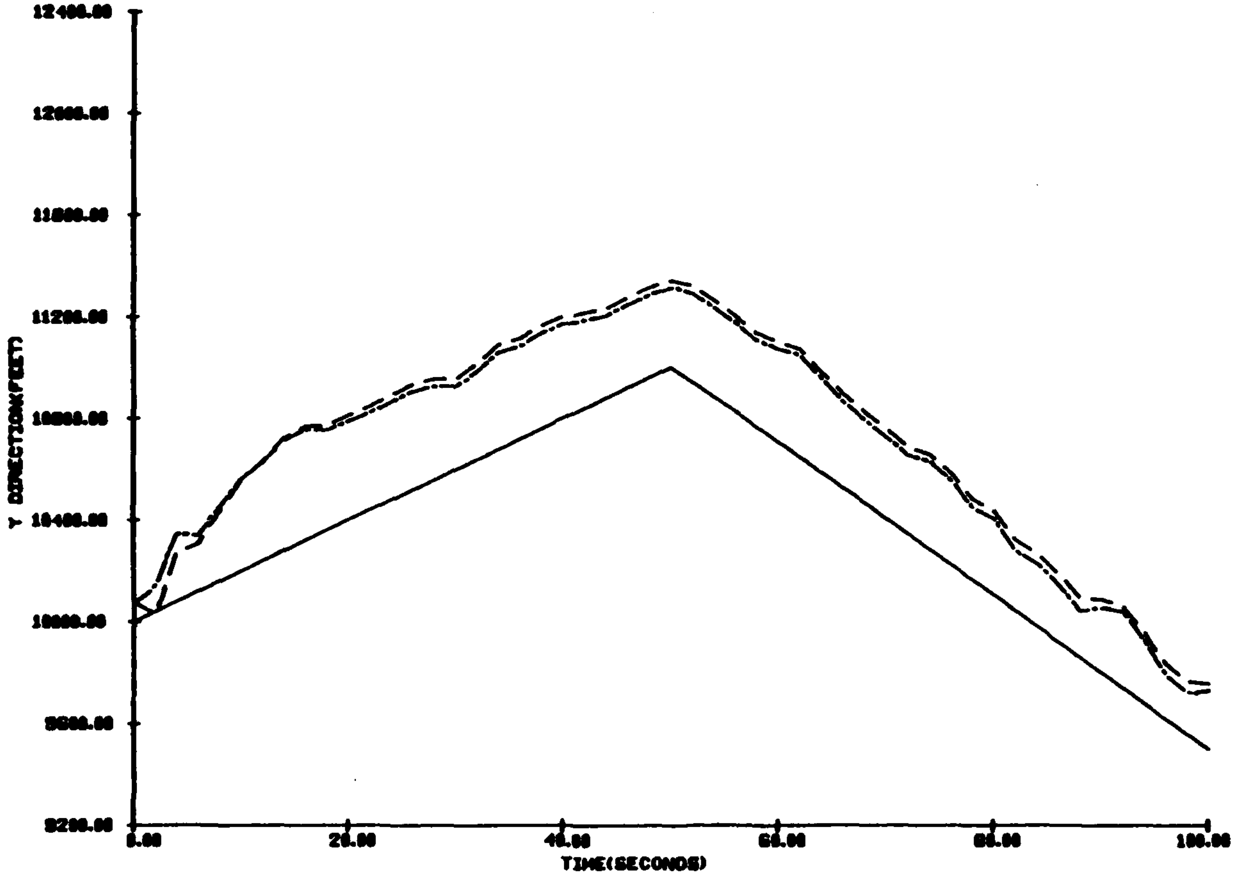

Figure 5.6 Ranges of Y-direction of (a) extended Kalman filter (-·-·-), (b) truncated second-order filter (---), and (c) reference trajectory (——) for maneuvering target

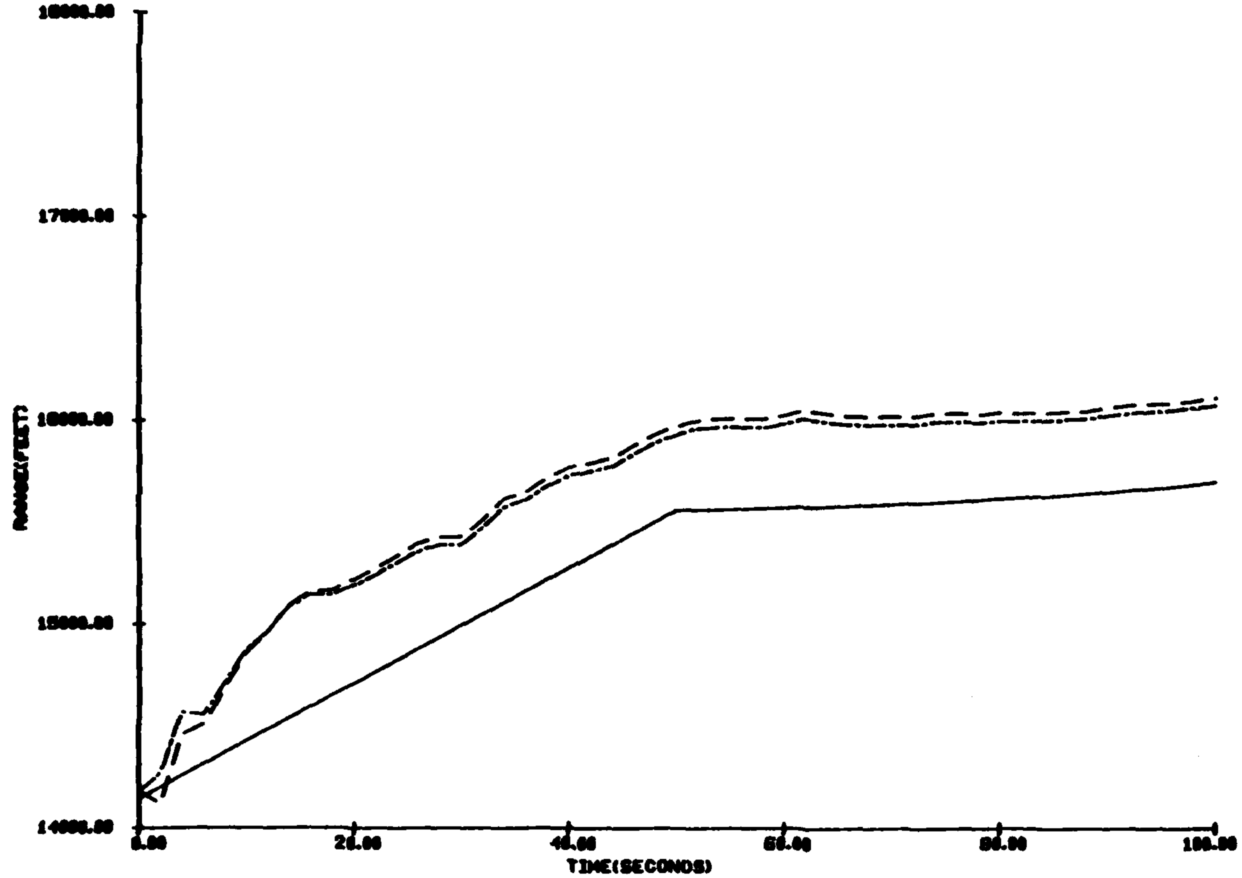

Figure 5.7 Ranges of (a) extended Kalman filter (-·-·-), (b) truncated second-order filter (---), and (c) reference trajectory (——) for maneuvering target

6. CONCLUSIONS

The contribution of this dissertation is in the presentation of the optimal and the suboptimal control for certain linear stochastic systems with known and unknown random coefficients in the Bellman-Hamilton-Jacobi equation and related algorithms.

To analyze the optimal control for a linear system with completely observable random coefficients, the proofs of existence of optimal control are based on the stochastic version of Bellman's principle of optimality and Kolodziej's results [29]. The suboptimal control for the class of certain stochastic control systems with unknown coefficients belongs to the coupled bilinear stochastic system is also studied here. The main assumption made is that a finite-dimensional filter, which is independent of the control and which gives a good estimate of the unobservable coefficients, can be found. Structure of this filter is used to construct the suboptimal control in equation (3-1). For the quadratic cost function, the explicit formulae describing the control law is derived. The gain of the linear-in-state feedback is a solution to certain nonlinear partial differential equations.

It is expected that the class of stochastic models, which satisfy the assumptions made here, includes many of these models for which there is no existing technique leading to strictly optimal results. The approximate stochastic model investigated with the stated assumptions produces a close optimal result. The measurement information about uncertain quantities use $E[z_t|Z_t, 0 \leq t \leq T]$, but this conditional estimate could not be used for

the exact stochastic model in equation (3-1) since its innovation process depends upon the feedback control law.

The numerical examples are provided to illustrate the previous results. The generation of the Wiener process is used for the random-walk theory with the pseudo-uniform random numbers and the pseudo-Gaussian random number $N(0,1)$. It turns out that the pseudo-Wiener process satisfies the general properties of the Wiener process and passes some proper random testing methods.

Numerical solution of the Riccati-like equation to the Cauchy problem suggests the need for more precise numerical methods to be simulated by digital computer. In general, simulation results of this type of nonlinear partial differential equation use excessive computer time.

For bad weather conditions of an aircraft landing system in wind gusts, the theoretical results of Chapter 3 illustrate the design procedure of the optimal control. It is found that the sub-optimal control-design technique to the stochastic-parameter linear system provides an effective method for choosing the appropriate stochastic models. Moreover, the simulation results of altitude of aircraft is almost the same as the exponential, linear-flare path for safe comfortable landing. The above method can also be applied to the control problems of longitudinal motion of an aircraft in gusty wind containing a dynamical feedback controller. It makes it possible to ensure the proper trajectories of the angle of attack and the orientation rate for optimal control law of the elevator control angle and the aileron control angle given a

reasonable cost function. The method proposed here can also be applied to unobservable stochastic systems if there exist the conditional Gaussian filter and simultaneous solution of certain nonlinear partial differential equations corresponding to given stochastic models.

To approximate stochastic models, the suboptimal control in equation (3-1) is a fruitful area for developing improved control policies. The suboptimal control has considerable complexity because of a linear function of observable states and a nonlinear function of uncertain random parameters. However, this author believes that suboptimal feedback control presents the most useful optimization result in stochastic control theory.

It describes the nonlinear target model for an optimal decision algorithm to decide between maneuvering and nonmaneuvering targets. The extended Kalman filter and the truncated second-order filter for implementation in anti-submarine warfare, target-motion analysis are developed and presented in Chapter 5. The target-motion equation for maneuvering given by (5-12) is modeled in the form of a bilinear stochastic system when the unknown accelerations are generated from the equation (5-12). The optimal estimate can produce the solution of (5-16) and (5-17) which have to be solved together. It may be possible to estimate adaptively the covariance $\Upsilon(t)$, which is dependent on the original states and a target maneuver. Preliminary work using realistic computer simulation for no maneuvering indicates that the extended Kalman filter and the truncated second-order filter accuracies agree to

within about 3 percent. Results shown in Figure 5.4 show that the truncated second-order filter is much better than the extended Kalman filter for the time duration of the first 15 seconds. After 15 seconds the extended Kalman filter performs better than the truncated second-order filter because the nonlinearities (5.3) decrease in significance.

This section discusses some special problems that have yet to be proven in the area of optimal control. For practical applications of known results in the area of optimal control, it is necessary to develop efficient numerical technique for solving nonlinear partial differential equations in (2-18) and (3-19) which have multi-dimensions.

It would be also interesting to study the problem of optimal control for stochastic dynamic systems with discrete random coefficients like Markov jump processes. For example if the target maneuvers, a detector has to estimate the unknown input accelerations of target motion. The maneuvering characteristic of the target may be described by the stochastic model provided. Also the finite-state, continuous-time, Markov process with weak interaction may be modeled as a singularly perturbed system, such as by queueing network models of computer systems which accentuates the need for reduced-order approximations of large-scale Markov chains. The theoretical results which are provided here may be applied to the singularly perturbed form to simplify the treatment of cost equations and decentralized algorithms in optimization problems.

The most important results desired are with respect to more

adequate mathematical models that can be used for synthesis of optimal control laws. The estimate $E[z_t | \mathcal{Y}_t]$ is suboptimal in a sense that the state variable x_t also includes information about uncertain quantities. However, the contruction of the conditional estimate is not available with respect to the σ-algebra $\mathcal{Z}_t$ because of dependence on the feedback control law.

Also, it would be useful to develop conditions of stability for bilinear stochastic systems governed by nonlinear Ito differential equations equipped with random coefficients which have incomplete state information but to make it available for appropriate feedback control.

REFERENCES

1. Fleming, W. H., "Optimal Control of Partially Observable Diffusions," SIAM J. Control 6, (1968). pp. 194-213.

2. Fleming, W. H., "The Cauchy Problem for Degenerate Parabolic Equations," J. Math. Mech. 13, (1964), pp. 987-1008.

3. Fleming, W. H., "Optimal Continuous-Parameter Stochastic Control," SIAM Review 11, (1969), pp. 470-509.

4. Fleming, W. H., and Rishel, R. W., Deterministic and Stochastic Optimal Control, (1975), Springer-Verlag, New York.

5. Kushner, H. J., "On the Dynamical Equations of Conditional Density Functions with Application to Optimal Stochastic Control Theory," J. Math Anal. Appl. 8, (1964), pp. 332-344.

6. Kushner, H. J., "On the Existence of Optimal Stochastic Controls," SIAM J. Control 3, (1966), pp. 463-474.

7. Wonham, W. M., "On the Separation Theorem of Stochastic Control," SIAM J. Control 6, (1968), pp. 312-316.

8. Wonham, W. M., "Optimal Stationary Control of Linear Systems with Dependent Noise," SIAM J. Control 5, (1967), pp. 486-500.

9. Wonham, W. M., "Random Differential Equation in Control Theory," in Probabilistic Methods in Applied Mathematics, Vol. II, (1970) ed. by Bharucha-Reid, New York Academic Press, pp. 131-242.

10. Balakrishnan, A. V., "Stochastic Control: A Function Space Approach," SIAM J. Control 10, (1972), pp. 285-297.

11. Balakrishnan, A. V., "Modeling and Identification: A Flight Control Application," in Theory and Applications of Variable Structure Systems, (1972), ed. by Mohler, R. R. and Ruberti, A., Academic Press New York, pp. 1-24.

12. Benés, V. E., "Existence of Optimal Stochastic Control Laws," SIAM J. Control 9, (1972), pp. 446-472.

13. Benés, V. E., "Existence of Optimal Strategies Based on Specified Information for a Class of Stochastic Decision Problems," SIAM J. Control 8, (1970), pp. 179-188.

14. Rishel, R. W., "Necessary and Sufficient Dynamic Programming Conditions for Continuous Time Stochastic Optimal Control," SIAM J. Control 8, (1970), pp. 559-571.

15. Rishel, R. W., "Optimality of Controls for System with Jump Markov Disturbances," SIAM J. Control 13, (1975), pp. 338-171.

16. Bismut, J. M., "Linear Quadratic Optimal Stochastic Equations with Random Coefficients," SIAM J. Control and Optimization 14, (1976) pp. 419-444.

17. Bismut, J. M., "On Optimal Control of Linear Stochastic Equations with Linear Quadratic Criterion," SIAM J. Control and Optimization 15, (1977), pp. 1-4.

18. Bismut, J. M., "An Introductory Approach to Duality in Optimal Stochastic Control," SIAM Review 20, (1978), pp. 62-78.

19. Davis, M. H. A., and Varaiya, "Dynamic Programming Conditions for Partially Observable Stochastic Systems," SIAM J. Control 11, (1973), pp. 226-261.

20. Davis, M. H. A., "On the Existence of Optimal Policies in Stochastic Control," SIAM J. Control 11, (1973), pp. 587-594.

21. Davis, M. H. A., "The Separation Principles in Stochastic Control via Girsanov Solutions," SIAM J. Control and Optimization 14, (1976), pp. 176-188.

22. Davis, M. H. A., "The Separation Theorem for Linear Systems with Quadratic Cost Functions," Publication 72/36 (Lecture notes), (1972), Departement of Computing and Control, Imperial College, London SW7 2BT.

23. Davis, M. H. A., "Martingale Methods in Stochastic Control," in Stochastic Control Theory and Stochastic Differential Systems, (1979), ed. by Kohlmann, K. and Vogel, W., Spring-Verlag, New York, pp. 85-117.

24. Eliott, R. J., "The Optimal Control of a Stochastic Control Systems," SIAM J. Control and OPtimization 15, (1977), pp. 756-778.

25. Eliott, R. J., "A Sufficient Condition for the Optimal Control of a Partially Observed Stochastic System," in Analysis and Optimzation of Stochastic Systems, (1979) ed. by Jacobs, D. L. R., Academic Press, New York, London.

26. Haussmann, U., "On the Existence of Moments of Stationary Linear Systems with Multiplicative Noise," SIAM J. Control 12, (1974), pp. 99-105.

27. Haussmann, U., "Optimal Stationary Control with State and Control Dependent Noise," SIAM J. Control 9, (1971), pp. 184-198.

28. Variya, P., "Optimal Control of a Partially Observed Stochastic System," Proc. Symp. Appl. Math., (1972), New York pp. 173-187.

29. Kolodziej, W. J., "Conditionally Gaussian Process in Stochastic Control Theory," (1980), Ph.D. Dissertation, Oregon State University, Corvallis, Oregon.

30. Mohler, R. R. and Kolodziej, W. J., "Optimal Control of a Class of Nonlinear Stochastic Systems," IEEE Trans. Auto. Cont. Vol. AC-26, (1981), No. 5, pp. 1048-1054.

31. Mohler, R. R. and Kolodziej, W. J., "An Overview of Stochastic Bilinear Control Processes," IEEE Trans. Systems, Man and Cybernetics, Vol. SMC-10, (1980), No. 12, pp. 913-918.

32. Leitmann, G., "Guaranteed Ultimate Boundness for a Class of Uncertain Linear Dynamic Systems," IEEE, Trans. Auto. Cont., Vol. AC-23, (1978), pp. 1109-1110.

33. Leitmann, G., "On the Efficacy of Nonlinear Control in Uncertain Linear Systems," ASME J. Dynamic System, Measurement & Control, Vol. 102, (1981), pp. 95-102.

34. Leitmann, G., "Stabilization of Dynamical Systems under Bounded Input Disturbances and Parameter Uncertainty," in Differential Games and Control Theory II, (1977), ed. by Roxin, E. O., et al., Marcel Decker, Inc., pp. 47-64.

35. Ahmed, N. U., "Stochastic Control on Hilbert Space for Linea Evolution Equations with Random Operator-Valued Coefficients," SIAM J. Control and Optimzation 19, (1981), No. 3, pp. 401-430.

36. Athans, M., Ku, R., and Gershwin, S. B., "The Uncertainty Threshold Principle: Some Fundamental Limitations of Optimal Decision Making under Dynamic Uncertainty," IEEE Trans. Auto. Contr., Vol. AC-22, (1977), pp. 491-495.

37. Ku, R. and Athans, M., "Further Results on the Uncertainty Threshold Principle," IEEE Trans. Auto. Contr., Vol. AC-22, (1977), pp. 866-868.

38. Bensoussan, A. and Viot, B., "Optimal Control of Stochastic Linear Distributed Parameter Systems," SIAM J. Control 13, (1975), pp. 904-906.

39. McLane, P., "Optimal Stochastic Control of Linear Systems with State-and Control Dependent Disturbances," IEEE Trans. Auto. Contr., Vol. AC-16, (1971), pp. 793-798.

40. McLane, P., "Linear OPtimal Stochastic Control using Instaneous Output Feedback," Int. J. Control, Vol. 13, (1971), pp. 383-396.

41. Bruni, C., Dipillo, G., and Koch, G., "Bilinear Systems: An Appealing Class of "Nearly Linear" System in Theory and Applications," IEEE Trans. Auto. Contr., Vol. AC-19 (1974), pp. 334-348.

42. Sincovec, R. F., and Madsen, N. K., "Solution for Nonlinear Partial Differential Equations," ACM Trans. on Math. Software, (1975), pp. 232-260.

43. Balakrishnan, A. V., Stochastic Filtering and Control, Optimization, (1981), Software, Inc., Los Angeles.

44. Grishin, V. P., "On a Calculation Method Related to a Process of Automatic Adaptation," Automatic Remote Control, (1963), No. 12, pp. 1502-1509.

45. Holley, W. E., and Bryson, A. E., "Wind Modeling and Lateral Control for Automatic Landing," AIAA J. Spacecraft and Rockets, (1977), Vol. 14, No. 2, pp. 65-72.

46. Dryden, H. L., "Turbulence Investiation at the National Bureau of Standards," Proc. 5th International Congress of Applied Mechanics, 1938, p. 365.

47. Singer, R. A., "Estimating Optimal Tracking Filter Performance for Manned Maneuvering Targets," IEEE Trans. Aerosp. Electron. Syst., Vol. AES-6, (1970), pp. 473-483.

48. Moose, R. L., Vanlandingham, H. F., and McCabe, D. H., "Modeling and Estimation for Tracking Maneuvering Targets," IEEE Trans. Aerosp. Electron. Syst., Vol. AES-15, (1979), pp. 448-456.

49. Gholson, N. H., and Moose R. L., "Maneuvering Target Tracking Using Adaptive State Estimation," IEEE Trans. Aerosp. Electron. Syst., Vol. AES-13, (1977), pp. 310-319.

50. Chan, Y. T., Hu, A. G. C., and Plant, J. B., "A Kalman Filter Based Tracking Scheme with Input Estimation," IEEE Trans. Aerosp. Electron. Syst., Vol. AES-15, (1979), pp. 237-244.

51. Chou, S. I., "Some Drawbacks of Extended Kalman Filtering in ASW Passive Angle Tracking," in Advanced in Passive Target Tracking. Moneterey, CA, (1977): Naval Post Graduate School, pp. 76-113.

52. Aidala, V. J., "Behavior of the Kalman Filter Applied to Bearings-Only Target Motion Tracking," ibid. as 51, pp. 1-30.

53. Kerr, K. P., "Determing Hydrodynamic Coefficients by Means of Slender Body Theory," Lockheed Missiles and Space Company, (1959), IAD/790.

54. Hess, J. L., "Calculation of Potential Flow about Bodies of Revolution Having Axes Perpendicular to the Free Stream Direction," Douglas Aircraft Company, Report No. ES 29812, (1960).

55. Jazwinski, A. H., Stochastic Processes and Filtering Theory, Academic Press, (1970), New York.

56. Thorp, J. S., "Optimal Tracking of Maneuvering Targets," IEEE Trans. Aerosp. Electron. Syst., Vol. AES-9, (1973), pp. 512-519.

57. Sworder, D. D., "Bayes' Controller with Memory for a Linear System with Jump Parameters," IEEE Trans. Auto. Contr., Vol. AC-17, (1972), pp. 119-112.

58. Sworder, D. D. and Robinson, V. G., "Feedback Regulators for Jump Parameter System with State and Control Dependent Transition Rates," IEEE Trans. Auto. Contr., Vol. AC-18, 1973, pp. 355-360.

59. Friedland, B., "Notes on Separate-Bias Estimation," IEEE Trans Auto. Contr., Vol. AC-23, (1978), pp. 735-738.

60. Mehra, R. K. and Peschon, J., "An Innovations Approach to Fault Detection and Diagnosis in Dynamic Systems," Automatica, Vol. 7, (1971), pp. 637-640.

61. Willsky, A. S. and Jones, H. L., "A Generalized Likelihood Ratio Approach to the Detection and Estimation of Jumps in Linear SYstems," IEEE Trans. Auto. Contr., Vol. AC-21, (1976), pp. 108-112.

62. Chan, Y. T., Plant, J. B. and Bottomley, J. R. T., "A Kalman Tracker with a Simple Input Estimator," IEEE Trans. Aerosp. Electron. Syst., Vol. AES-18, (1980), 235-240.

63. Davis M, H. A., "The Application of Nonlinear Filtering to Fault Detection in Linear Systems," IEEE Trans. Auto. Contr., Vol. AC-20, (1975), pp. 257-259.

64. Liptser, R. S., and Shiryayev, A. N., Statistics of Random Processes I: General Theory, (1977), Springer-Verlag, New York.

65. Liptser, R. S., and Shiryayev, A. N., Statistics of Random Processes II: Applications, (1978), Springer-Verlag, New York.

66. Mohler, R. R., Bilinear Control Processes, (1973), Academic Press, New York.

67. Mohler, R. R., Barton, C. F., and Hsu, C. S., "T and B Cell Models in the Immune System," in Theoretical Immunology, ed. by Bell, G., et al., (1978), Marcel Dekker, New York.

68. Mohler, R. R., and Frick, P. A., "Bilinear Demographic Control Process," Int. J. Policy Anal. Inform. Syst., Vol. 2, (1979), pp. 57-70.

69. Kallianpur, G., Stochastic Filtering Theory, (1980), Springer-Verlag, New York.

70. Feller, W., An Introduction to Probability Theory and its Application, 3rd ed., (1968) Vol. 1, John Wiley & Sons, Inc., New York.

71. Gear, C. W., Numerical Initial Value Problems in Ordinary Differential Equations, (1971), Prentice Hall, Englewood Cliffs, N.J..

72. Hindmarsh, A. C., "Gear b: Solution of Ordinary Differential Equation Having Bounded Jacobian," Rep. UCID-30059, (1972), Lawrence Livermore Lab., Livermore, CA.

73. Hindmarsh, A. C., "Gear: Ordinary Differential Equation Solver," Rep. UCID-30001, (1972), Lawrence Livermore Lab., Livermore, CA.

74. Massey, F. J. Jr., "The Kolmogorov-Smirnov Test for Goodness of Fit," J. Amer. Stat. Assn., (1951), 46, pp. 68-78.

75. Frick, P. A., "On Walsh Noise: Its Properties and Use in Dynamic Stochastic Systems," IEEE Trans. on Sys., Man, and Cyber., Vol. SMC-9, (1979), No. 8, pp. 411-419.

76. Davis, M. H. A., Linear Estimation and Stochastic Control, John Wiley & Sons, Inc., (1977), New York.

77. Taylor, H. M., A First Course in Stochastic Process, (1975), Academic Press.

78. Etkin, B., Dynamics of Atmospheric Flight, (1972), John Wiley and Sons, Inc., New York, pp. 158-195.

79. Bass, R. D., Norum, V. D., and Swartz, L., "Optimal Multichannel Nonlinear Filtering," J. Math. Anal. Appl., Vol. 16, (1966), pp. 152-164.

80. Henriksen, R., "The Truncated Second-Order Nonlinear Filter Revisited," IEEE Trans. Auto. Contr., Vol. AC-27, (1982), pp. 247-251.

81. Mendel, J. M., and Washburn, H. D., "Multistage Estimation of Bias States," in Proc. (1976) IEEE Conf. on Decision and Control, pp. 629-630.

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